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Coupled-Mode Flutter of an Elastic Plate in a Gas Flow with a Boundary Layer

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Abstract—The stability of an elastic plate in a supersonic gas flow is considered in the presence of a boundary layer formed on the surface of the plate. The problem is solved in two statements. In the first statement, the plate is of large but finite length, and a coupled-mode type of flutter is examined (the effect of the boundary layer on another, single-mode, type of flutter has been studied earlier). In the second statement, the plate is assumed to be infinite, and the character of its instability (absolute or convective) is analyzed. In both cases, the instability is determined by a branch point of the roots of the dispersion equation, and the mathematical analysis is the same. It is proved that instability in a uniform gas flow is weakened by a boundary layer but cannot be suppressed completely, while in the case of a stable plate in a uniform flow the boundary layer leads to the destabilization of the plate.

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1. INTRODUCTION

In the classical theory of hydrodynamic stability, the stability of a fluid flow along nondeformable surfaces is usually considered. The need for the stabilization of laminar flows stimulated a number of studies on the stability of flows on flexible surfaces [1–5]. It was shown that the elastic and damping properties of a surface qualitatively change the shape of the neutral curve, as well as may change the character of instability from convective to absolute and, for a certain combination of these properties, may delay the laminar–turbulent transition. In the case of supersonic gas flows on a flexible surface, another type of instability arises—panel flutter—which is more hazardous due to the appearance of intense vibrations of a structure than due to the turbulization of the flow. This phenomenon is well known in aviation and has been studied by numerous authors since the 1950s. Until recently, only one type of panel flutter was known, the so-called *coupled-mode flutter*, which is attributed to the coupling of two eigenmodes of a plate through a gas flow. In [6], on the basis of the asymptotic method of [7, 8], it was proved that in the case of wide plates there exists another type of panel flutter, the so-called *single-mode flutter*, which arises for small supersonic velocities of the flow and which had remained unnoticed due to oversimplifications in the aerodynamic part of the problem. Later, the existence of single-mode flutter was confirmed numerically [9] and experimentally [10].

In the overwhelming majority of studies of panel flutter, the boundary layer formed on the surface has not been taken into consideration; in a few studies devoted to the effect of the boundary layer [11–13], the authors considered only specific profiles of velocity and temperature, because these studies were devoted to modeling experiments [14, 15]. The case of an arbitrary profile determined from the external flow problem was analyzed by the asymptotic method of [7] in [16] for single-mode flutter of a plate. In the present paper, we consider the effect of the boundary layer on the coupled-mode flutter.

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According to [6], the coupled-mode flutter of a plate of finite width is associated with a branch point of the roots of the dispersion equation. It is known that such branch points are also responsible for the absolute instability of an infinite system, i.e., in the case in question, for the instability of an infinite plate in a gas flow. Therefore, all the results obtained in this paper for the coupled-mode flutter of plates of finite size extend automatically to the absolute instability of an infinite plate.

2. THE MAIN ASSUMPTIONS AND THE DISPERSION EQUATION

The statement of the problem and the assumptions are similar to those used in [16]. In the two-dimensional statement, we consider the stability of an elastic plate that is stretched with a given tensile force and is subjected on one side to a plane-parallel flow of a viscous gas (the growth of the boundary layer is neglected), as is shown in Fig. 1. The plate has a finite width and is embedded in an infinite nondeformable plane $z = 0$. The gas flow occupies the domain $z > 0$, and the unperturbed velocity u_0 and temperature T_0 are assumed to be given functions of z and, generally speaking, should be determined from the problem of a flow of a flight vehicle or another airframe whose skin panel is the plate under consideration.

As a model of a plate, we take the classical Kirchhoff–Love model. The equations for the perturbations of a gas are given by the system of linearized Navier–Stokes equations, which is similar to the Orr–Sommerfeld equation for an incompressible fluid. In the present study, just as in [16], we assume that the Reynolds number R is large; then the solutions of the indicated system can be approximated by the solutions of the limit inviscid equation for $R \to \infty$, i.e., of the compressible Rayleigh equation. We also assume that the perturbation wavelengths are much greater than the thickness δ of the boundary layer.

Let us make two remarks concerning the applicability of these assumptions. First, laminar boundary layers in supersonic flows are often observed experimentally for Reynolds numbers (based on the boundary layer thickness) of the order of up to 10^5, i.e., for R large enough for perturbations to be describable in the inviscid approximation. In particular, such layers are observed in flows with pressure gradient, surface cooling, or flow suction through the surface [17, Ch. 5]. If the boundary layer is turbulent but the characteristic pulsation frequencies are greater than the frequency of growing vibrations of the plate, then the equations of laminar flow can be used as the first approximation with appropriate averaged profiles of velocity and temperature.

Second, the solutions of a viscous system are approximated as $R \to \infty$ by the solutions of the Rayleigh equation not everywhere but only in a sector of the complex plane z with angle $4\pi/3$ and vertex $z = z_c$ at which $u_0(z_c) = c$; here $c = \omega/k$ is the phase velocity (Fig. 2) [18]. In the shaded sector, the solution of the viscous system has the Wentzel–Kramers–Brillouin (WKB) character, i.e., $u(z) \sim g(z)\exp(\sqrt{iR}f(z))$. This property leads to "Lin's rule": if $\operatorname{Im} z_c \le 0$ and the WKB sector occupies a part of the real half-axis $z \ge 0$ not containing the point $z = 0$, then the inviscid solution can be analytically continued from the point $z = 0$ (where boundary conditions are specified) to $z = +\infty$ (where boundary conditions are also specified) only by going out to the complex plane and bypassing the point z_c from below. If the real half-axis $z \ge 0$ does not belong to the WKB sector (for example, $\operatorname{Im} z_c > 0$), then the inviscid solution can be continued analytically along the real axis z. If the point $z = 0$ itself falls into the WKB sector, then the inviscid approximation cannot

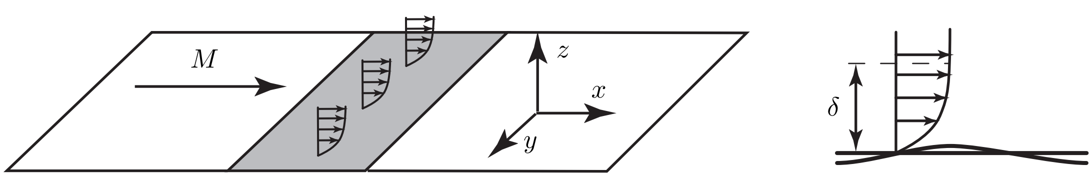

Fig. 1. Plate in a gas flow with a boundary layer.

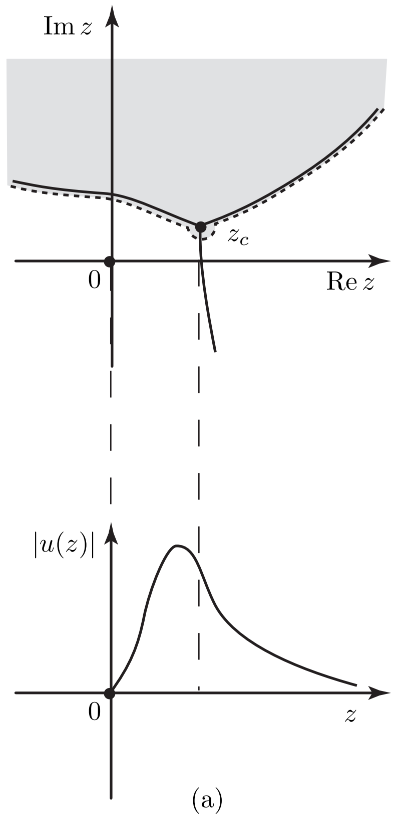

Fig. 2. Upper parts of the figure: Stokes lines and the domain of exponential behavior of the viscous solution when z_c lies (a) above and (b) below the real axis. Lower parts of the figure: visible behavior of perturbations for "physical" real z in both cases.

be applied, because the boundary conditions cannot be correctly satisfied at $z = 0$. We will assume that $u_0'(z) > 0$ throughout the flow; then, if z_c lies in some neighborhood of the real half-axis $z \geq 0$, the position of z_c in the complex plane z qualitatively coincides with the position of c in its own complex plane.

Under these assumptions, a dispersion equation was derived in [16] for the perturbations of an infinite coupled plate–flow system that depend on x and t as $e^{i(kx-\omega t)}$:

$$\mathcal{D}(k,\omega) = \left(Dk^4 + M_{\mathrm{w}}^2 k^2 - \omega^2\right) - \mu\left(\left(\frac{(M_\infty k - \omega)^2}{\sqrt{k^2 - (M_\infty k - \omega)^2}}\right)^{-1} + \delta\left(\int\limits_0^1 \frac{T_0(\eta)\,d\eta}{(u_0(\eta) - c)^2} - 1\right)\right)^{-1} = 0. \tag{2.1}$$

Here k and ω are the wave number and frequency of the wave; the dimensionless parameters D and M_{w} are stiffness and a parameter characterizing the tension of the plate; and M_∞, μ, and δ are the Mach number of the flow outside the boundary layer, the ratio of the flow density to the density of the plate material, and the thickness of the boundary layer (normalized by the plate thickness), respectively. The functions $u_0(\eta)$ and $T_0(\eta)$ describe the dependence of the velocity and temperature of the unperturbed flow on the vertical coordinate $\eta = z/\delta$. The parameter μ is assumed to be small, because it is of the order of 10^{-5} to 10^{-3} in applications.

We carry out the analysis by the asymptotic method of global instability [7]. The roots $k_j(\omega)$ of the dispersion equation, numbered in decreasing order of $\operatorname{Im} k_j$ for $\operatorname{Im}\omega \gg 1$, are divided into two groups: the first group contains roots such that $\operatorname{Im} k_j(\omega) > 0$ as $\operatorname{Im}\omega \to +\infty$, $j = 1,\dots,s$, and the second group contains roots such that $\operatorname{Im} k_j(\omega) < 0$ as $\operatorname{Im}\omega \to +\infty$, $j = s+1,\dots,N$. The first group describes waves traveling from left to right, and the second group, from right to left. In [7], Kulikovskii obtained an equation describing the limit set of eigenvalues of a finite system (in our case, a plate) of width L as $L \to \infty$ in the form

$$\min_{j=1,\dots,s} \operatorname{Im} k_j(\omega) = \max_{j=s+1,\dots,N} \operatorname{Im} k_j(\omega).$$

This equation does not depend on the boundary conditions specified at the ends of the plate (because they include the solutions of the dispersion equation for an infinite plate) and describes a curve Ω (in the complex plane ω) near which the eigenvalues of a finite plate in a gas flow are concentrated. If a part of this curve lies in the domain $\operatorname{Im}\omega > 0$, then a plate of large but finite size is unstable, because there exists an eigenfrequency of the plate in the neighborhood of this part of Ω.

For $\delta = 0$, there are four roots of (2.1): k_1 and k_2 belong to the first group, while k_3 and k_4, to the second. The curve Ω consists of two parts [6]. The first (Ω_1) is defined by the equation $\operatorname{Im} k_2(\omega) = \operatorname{Im} k_3(\omega)$ and corresponds to single-mode flutter of the plate. The second (Ω_2) is defined by the equation $\operatorname{Im} k_2(\omega) = \operatorname{Im} k_4(\omega)$ and corresponds to coupled-mode flutter. It ends with the branch point $k_2(\omega) = k_4(\omega)$, at which $\operatorname{Im}\omega$ attains its maximum. This branch point also leads to the absolute instability of the infinite plate. Below we study the effect of the boundary layer on this branch point.

3. SIMPLIFICATION OF THE DISPERSION EQUATION

In the absence of a boundary layer ($\delta = 0$), the branch point of $k(\omega)$ responsible for coupled-mode flutter of a finite plate and the absolute instability of an infinite plate satisfies the condition $|\omega| \ll |k|$, namely, $k \sim \mu^{1/3}$ and $\omega \sim \mu^{2/3}$ [6]. Below we will show that the orders of magnitude remain the same for $\delta \neq 0$; this allows us to simplify the dispersion equation (2.1). First, we can neglect ω compared with $M_\infty k$ in the first term in the parentheses multiplied by μ; i.e., we can replace

$$\frac{(M_\infty k - \omega)^2}{\sqrt{k^2 - (M_\infty k - \omega)^2}} \quad \to \quad -i\frac{M_\infty^2}{\sqrt{M_\infty^2 - 1}}k.$$

The choice of the branch of the square root was substantiated in [6].

Second, we can simplify the integral term. To this end, we notice that $c = \omega/k \sim \mu^{1/3}$ at the branch point in question. In view of the no-slip condition for the unperturbed flow, we have $u_0(0) = 0$; hence the point η_c at which $u_0(\eta_c) = c$ can be approximately expressed as follows: $\eta_c \approx c/u_0'(0) \sim \mu^{1/3}$. This point is a pole of the integrand, and it lies in a small neighborhood of the endpoint $\eta = 0$ of the integration interval. Hence, the main contribution to the integral is given by this neighborhood. To calculate the integral, we expand the profiles of the boundary layer in the Taylor series in the neighborhood of $\eta = \eta_c$:

$$T_0(\xi) = T_{00} + T_{01}\xi + \dots, \qquad u_0(\xi) = c + u_{01}\xi + \frac{1}{2}u_{02}\xi^2 + \dots, \qquad \xi = \eta - \eta_c.$$

Here T_{0n} and u_{0n} are the nth derivatives of the functions at the critical point. We have

$$\frac{T_0(\xi)}{(u_0(\xi) - c)^2} = \frac{T_{00} + T_{01}\xi + \dots}{\left(u_{01}\xi + \frac{1}{2}u_{02}\xi^2 + \dots\right)^2} = \frac{T_{00}}{u_{01}^2}\frac{1}{\xi^2} + \frac{1}{u_{01}^2}\left(T_{01} - T_{00}\frac{u_{02}}{u_{01}}\right)\frac{1}{\xi} + \text{reg. terms.} \tag{3.1}$$

After the integration of (3.1), the leading term of the integral in the dispersion equation becomes

$$\int\limits_{-\eta_c}^{1-\eta_c} \frac{T_{00}}{u_{01}^2}\frac{1}{\xi^2}\,d\xi = -\frac{T_{00}}{u_{01}^2}\frac{1}{\xi}\bigg|_{-\eta_c}^{1-\eta_c} = -\frac{T_{00}(\eta_c)}{u_{01}^2(\eta_c)}\left(\frac{1}{1-\eta_c} + \frac{1}{\eta_c}\right) \to -\frac{T_0(0)}{u_0'(0)}\frac{1}{c}$$

as $c \to 0$.

Thus, for small μ, the dispersion equation (2.1) can be simplified as follows:

$$\mathcal{D}(k,\omega) = \left(Dk^4 + M_{\mathrm{w}}^2 k^2 - \omega^2\right) - \mu\left(i\frac{\sqrt{M_\infty^2-1}}{M_\infty^2}\frac{1}{k} - \delta\frac{T_0(0)}{u_0'(0)}\frac{k}{\omega}\right)^{-1} = 0. \tag{3.2}$$

It is easy to notice that for the orders of k and ω indicated above, both terms in the parentheses multiplied by μ are of the same order $\mu^{-1/3}$.

Next, it is convenient to rewrite (3.2) as

$$\mathcal{D}(k,\omega) = \left(Dk^4 + M_{\mathrm{w}}^2 k^2 - \omega^2\right) - \frac{\mu\omega k}{ia\omega - \delta b k^2} = 0, \tag{3.3}$$

where, for brevity, we introduced the parameters

$$a = \frac{\sqrt{M_\infty^2-1}}{M_\infty^2} > 0 \qquad \text{and} \qquad b = \frac{T_0(0)}{u_0'(0)} > 0.$$

The saddle points of the solutions (3.3) $\omega(k)$ (which are the branch points of $k(\omega)$) are given by the condition $\partial\mathcal{D}/\partial k = 0$; i.e.,

$$4Dk^3 + 2M_{\mathrm{w}}^2 k - \frac{\mu\omega(ia\omega + \delta b k^2)}{(ia\omega - \delta b k^2)^2} = 0. \tag{3.4}$$

To calculate the branch points, instead of (3.3) we will use an equivalent equation obtained by multiplying (3.3) by 4 and subtracting (3.4) multiplied by k from it:

$$2M_{\mathrm{w}}^2 k^2 - 4\omega^2 + \frac{\mu\omega k(-3ia\omega + 5\delta b k^2)}{(ia\omega - \delta b k^2)^2} = 0. \tag{3.5}$$

Thus, the branch points of $k(\omega)$ are the solutions of system (3.4), (3.5). In general, not all these solutions satisfy the condition that the point $z = 0$ does not belong to the WKB sector (Fig. 2); i.e., some of these solutions require the gas viscosity to be taken into account in the equations for perturbations for arbitrarily large Reynolds numbers. One cannot exclude the possibility that such branch points lead to absolute instability, as in [4], where the instability is attributed to the coalescence of the Tollmien–Schlichting wave generated by the boundary layer with a wave generated by the plate. However, because of the relatively slow growth of the Tollmien–Schlichting waves compared with the growth of vibrations due to flutter, we can assume that this absolute instability is also weak compared to flutter. Therefore, in this paper we restrict ourselves to the partial problem: the analysis of the effect of the boundary layer on a specific branch point at which the waves generated by the plate coalesce in a uniform flow and which is responsible for the coupled-mode flutter of a finite plate and for the absolute instability of an infinite plate in a uniform gas flow. Below we will show that this branch point satisfies the condition $\operatorname{Im} c > 0$; i.e., the study of this point in the inviscid approximation is correct.

We will select branch points responsible for instability by means of “topographic positioning.” Namely, taking some specific parameters of the problem and constructing numerically the level lines of $\operatorname{Im}\omega(k)$ or $\operatorname{Re}\omega(k)$, we mark the saddle points at which branches from different groups coalesce. Next, changing the parameters of the problem, we only need to trace the position of the marked branch points (which can often be done analytically), because the topology of the level lines does not change under moderate variation of the parameters. When there occurs a bifurcation of the branch points and the associated change in the topology of the level lines, we should carry out the topographic positioning anew, and again only for a specific set of parameters. As the parameters for the topographic positioning, we take

$$D = 23.9, \qquad M_\infty = 1.5, \qquad \mu = 0.00012, \tag{3.6}$$

which correspond to a steel plate in an air flow at an altitude of 3 km.

Let us discuss the partition of the roots of (3.3) into groups. For $\delta = 0$, there are four roots $k(\omega)$, with two roots in each group: $\operatorname{Im} k_{1,2}(\omega) \to +\infty$ and $\operatorname{Im} k_{3,4}(\omega) \to -\infty$ as $\operatorname{Im}\omega \to +\infty$. For large $|\omega|$, these roots are close to the roots of (3.3) with $\mu = 0$, i.e., to the roots of the dispersion equation of a plate in a vacuum. For $\delta \neq 0$, two more roots arise; as $\operatorname{Im}\omega \to +\infty$, one of them tends to $+i\infty$, and the other, to $-i\infty$; for large $|\omega|$, these roots are close to the zeros of the denominator of the fraction in (3.3). We denote the first root by $k_{\mathrm{bl}+}(\omega)$ and the second by $k_{\mathrm{bl}-}(\omega)$. Now, each group consists of three roots: k_1, k_2, $k_{\mathrm{bl}+}$ and k_3, k_4, $k_{\mathrm{bl}-}$.

It suffices to analyze the position of the branch points in the half-plane $\operatorname{Re}\omega \geq 0$. Indeed, for the frequency ω reflected with respect to the imaginary axis, the corresponding solution $k(\omega)$ of (3.3) is also reflected with respect to the imaginary axis. Physically, these reflected solutions describe the same wave.

4. PLATE IN A UNIFORM FLOW

For $\delta = 0$, system (3.4), (3.5) reduces to

$$\begin{gathered} 4Dk^3 + 2M_{\mathrm{w}}^2 k + \frac{i\mu}{a} = 0, \\ 2M_{\mathrm{w}}^2 k^2 - 4\omega^2 + 3\frac{i\mu}{a} k = 0, \end{gathered} \tag{4.1}$$

which was investigated analytically in [6]. Here we present the results of these investigations.

For $M_{\mathrm{w}} = 0$ (absence of tension), there are three branch points in the plane k that correspond to $\operatorname{Re}\omega \geq 0$; the topographic positioning is shown in Fig. 3a. The first point

$$k^* = \Big(\frac{\mu}{4Da}\Big)^{1/3} e^{-i\pi/6}, \qquad \omega^* = \frac{\sqrt{3}}{2}\Big(\frac{\mu}{a}\Big)^{2/3} \frac{1}{(4D)^{1/6}} e^{i\pi/6}$$

is responsible for the global instability of a plate of finite length, because the branches k_2 and k_4 from different groups coalesce at this point. This global instability causes the coupled-mode flutter of a plate in a uniform flow. This point is also responsible for the absolute instability of an infinite plate, because a deformed integration contour passes through it. The second point

$$k^{**} = \Big(\frac{\mu}{4Da}\Big)^{1/3} e^{-5i\pi/6}, \qquad \omega^{**} = \frac{\sqrt{3}}{2}\Big(\frac{\mu}{a}\Big)^{2/3} \frac{1}{(4D)^{1/6}} e^{-i\pi/6}$$

results from the coalescence of the branch k_3 with the branch formed at the first point. Obviously, this point does not lead to instability. The third point

$$k^{***} = \Big(\frac{\mu}{4Da}\Big)^{1/3} i, \qquad \omega^{***} = \pm\frac{\sqrt{3}}{2}\Big(\frac{\mu}{a}\Big)^{2/3} \frac{1}{(4D)^{1/6}} i$$

results from the coalescence of the branches k_1 and k_2 and has no relation to instability. Indeed, in the case of an infinite plate, the integration contour does not pass through this point, while in the case of a finite plate, branches from the same group of waves coalesce at this point.

As M_{w} increases within the range

$$M_{\mathrm{w}} < M_{\mathrm{w}}^{\mathrm{cr}} = \sqrt{\frac{3}{2}}\Big(\frac{\mu}{a}\Big)^{1/3} D^{1/6},$$

the frequencies ω^* and ω^{**} approach the real axis but remain in the same quadrants of the complex plane as for $M_{\mathrm{w}} = 0$.

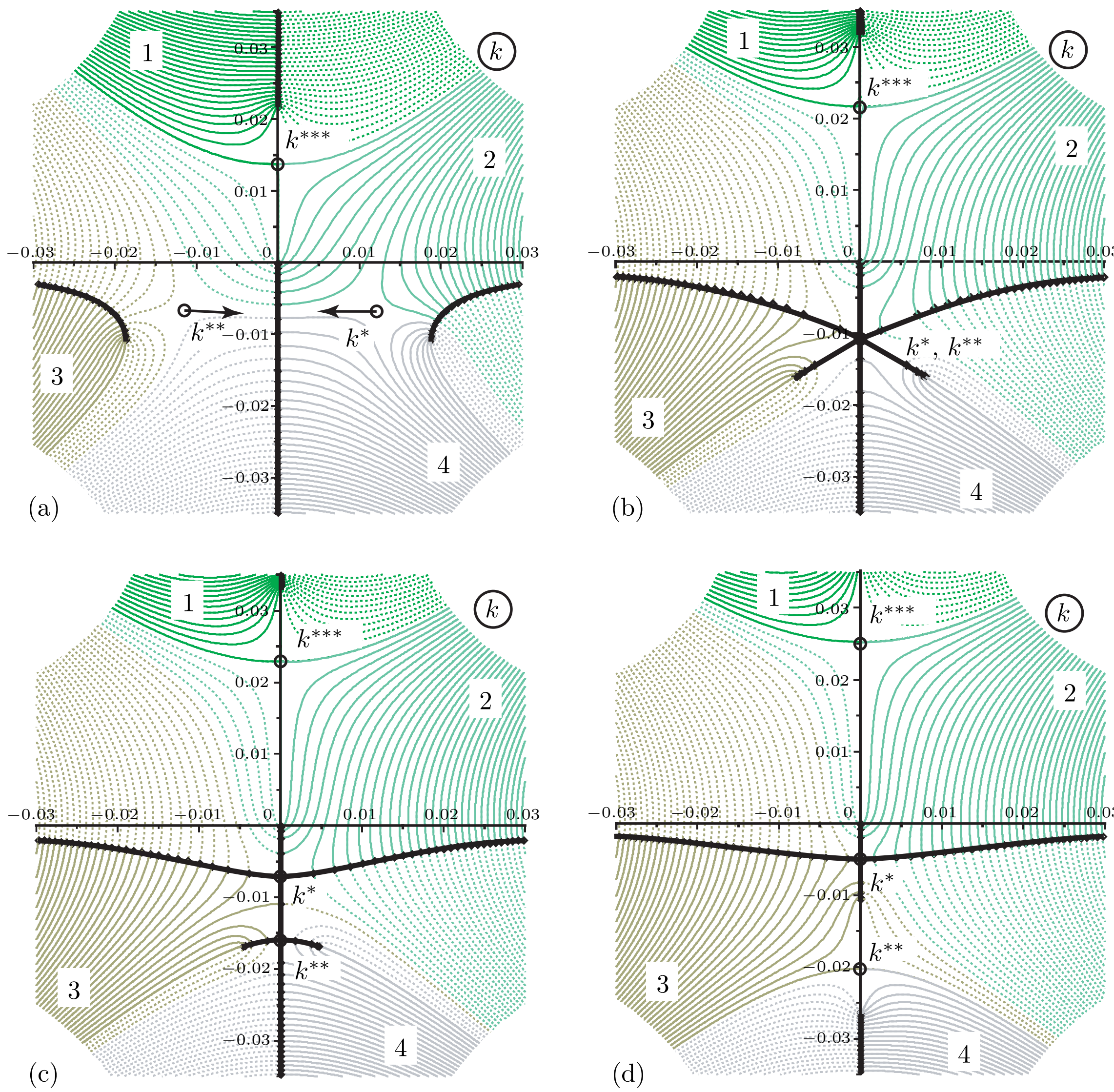

Fig. 3. Level lines $\mathrm{Re}\,\omega(k) = \mathrm{const} \geq 0$ in the complex plane k for parameters (3.6): (a) $M_\mathrm{w} < M_\mathrm{w}^\mathrm{cr}$; (b) $M_\mathrm{w} = M_\mathrm{w}^\mathrm{cr}$; (c) $M_\mathrm{w} > M_\mathrm{w}^\mathrm{cr}$ and ω^{**} is real; and (d) $M_\mathrm{w} > M_\mathrm{w}^\mathrm{cr}$ and ω^{**} is purely imaginary. Solid lines are the images of $\mathrm{Im}\,\omega > 0$, and dashed lines are the images of $\mathrm{Im}\,\omega < 0$. Heavy lines are the level lines $\mathrm{Im}\,\omega = 0$, $\mathrm{Re}\,\omega \geq 0$. The numbers show the indices of the branches that map the sector $\mathrm{Re}\,\omega > 0$, $\mathrm{Im}\,\omega > 0$ to the domain containing this index. The branch points of $k(\omega)$ are shown by circles, and the directions of motion of these points as M_w increases are shown by arrows.

For $M_\mathrm{w} = M_\mathrm{w}^\mathrm{cr}$, a bifurcation occurs (Fig. 3b): the first and second branch points coalesce on the real axis,

$$\omega^* = \omega^{**} = \frac{\sqrt{3}}{4}\left(\frac{\mu}{a}\right)^{2/3}\frac{1}{D^{1/6}},$$

and for $M_\mathrm{w} > M_\mathrm{w}^\mathrm{cr}$ these points remain real. In the plane k, the coalescence occurs on the imaginary axis:

$$k^* = k^{**} = -\frac{i}{2}\left(\frac{\mu}{Da}\right)^{1/3},$$

and after the coalescence both points remain purely imaginary. The topographic positioning (Fig. 3c) shows that the point corresponding to the smaller $|k|$ is responsible for stability; we will denote it by k^* as before. Since the corresponding frequency is real, for $M_{\mathrm{w}} > M_{\mathrm{w}}^{\mathrm{cr}}$ the instability of an infinite plate becomes convective, while coupled-mode flutter in a finite plate disappears (however, the possibility of single-mode flutter remains [6]).

A further increase in M_{w} gives rise to another bifurcation, the motion of ω^{**} along the real axis to 0 and coalescence with the branch point reflected with respect to the imaginary axis (this bifurcation is only connected with the second equation in (4.1); i.e., it occurs only in the plane ω). After that, ω^{**} moves along the imaginary axis and remains on it for arbitrarily large M_{w}. This restructuring is illustrated in Fig. 3d.

The third branch point ω^{***} does not interact with the other branch points for any M_{w} and, as before, has no relation to the problem of stability.

5. PLATE IN A BOUNDARY LAYER IN THE ABSENCE OF TENSION

First, consider the case of $M_{\mathrm{w}} = 0$; in this case, the position of the branch point can be treated analytically. We carry over the term with μ to the right-hand side in (3.4) and (3.5) and divide one equation by the other. Then, k and ω appear in the equation obtained only as the combination $\lambda = k^2/(i\omega)$:

$$D\lambda^2 = \frac{a + \delta b\lambda}{3a - 5\delta b\lambda},$$

or

$$G(\lambda) = 5D\delta b\lambda^3 - 3Da\lambda^2 + \delta b\lambda + a = 0. \tag{5.1}$$

As soon as this cubic equation is solved, we can easily determine ω and k. To this end, we rewrite (3.5) as

$$4\frac{\omega^2}{k} = \mu i\beta(\lambda), \qquad \beta(\lambda) = \frac{3a - 5\delta b\lambda}{(a - \delta b\lambda)^2}.$$

Now, solving the system

$$4\frac{\omega^2}{k} = \mu i\beta(\lambda), \qquad \frac{k^2}{\omega} = i\lambda,$$

we finally obtain the branch points:

$$\omega = \left(\frac{-i\mu^2\lambda\beta^2(\lambda)}{16}\right)^{1/3}, \qquad k = -\frac{4i\omega^2}{\mu\beta(\lambda)}.$$

In the expression for ω, for every λ, three values of the cubic root give three different branch points. Since (5.1) has three solutions λ, there are nine different branch points in total. All of them satisfy the condition $k \sim \mu^{1/3}$, $\omega \sim \mu^{2/3}$; hence, the simplifications made in the dispersion equation for small μ are correct.

Let us consider the solutions of (5.1) and select the branch points responsible for instability. To this end, it is convenient to analyze the solutions of (5.1) graphically. For $\delta = 0$, there are two real solutions of (5.1): $\lambda_1 < 0$ and $\lambda_2 > 0$ (Fig. 4). Among the frequencies ω, of interest are those that lie in the right half-plane. The solution λ_1 gives frequencies ω^* and ω^{***}, the former being responsible for the coupled-mode flutter of a plate of finite length and for the absolute instability of an infinite plate, and the latter having no relation to instability. The solution λ_2 gives frequencies ω^{**}, which has a negative imaginary part, and ω^{***}.

For small $\delta \neq 0$, there appears another solution $\lambda_3 > 0$ (Fig. 4), with $\lambda_3 \sim 1/\delta$ as $\delta \to 0$. This solution gives two branch points: at one point, the branches k_1 and $k_{\mathrm{bl}+}$ coalesce, and at the

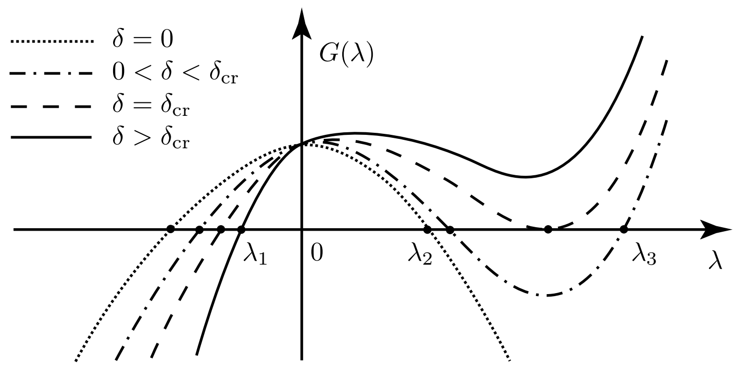

Fig. 4. Graph of $G(\lambda)$ for various values of δ.

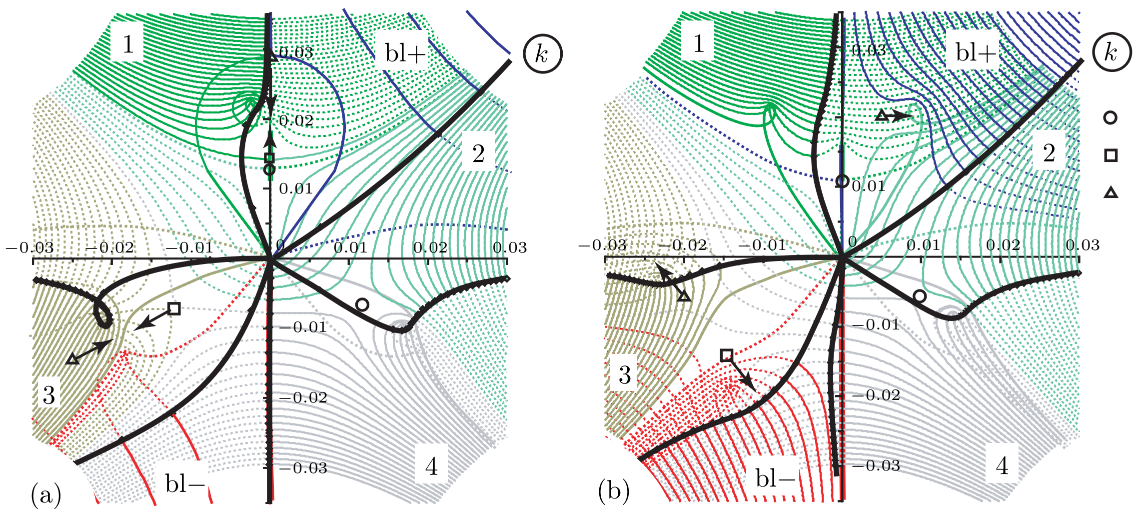

Fig. 5. Level lines $\mathrm{Re}\,\omega(k) = \mathrm{const} \geq 0$ in the complex plane k for parameters (3.6) and $M_{\mathrm{w}} = 0$: (a) before bifurcation, $\delta = 0.2$, and (b) after bifurcation, $\delta = 1.2$. Solid lines represent the images of $\mathrm{Im}\,\omega > 0$, and the dashed lines, the images of $\mathrm{Im}\,\omega < 0$. Heavy lines are the level lines $\mathrm{Im}\,\omega = 0$, $\mathrm{Re}\,\omega \geq 0$. At the perimeter, the indices of those branches are shown that map the quadrant $\mathrm{Re}\,\omega > 0$, $\mathrm{Im}\,\omega > 0$ to the domain containing this index. The branch points of $k(\omega)$ corresponding to λ_1, λ_2, and λ_3 are shown by circles, squares, and triangles, respectively; the arrows indicate the directions of motion of these points as δ increases.

other, the branches k_3 and $k_{\mathrm{bl}-}$ coalesce, which is established by means of topographic positioning (Fig. 5a). One can see that these points have no relation to instability.

For all three solutions λ_j, as long as they remain real, we have $\beta(\lambda_j) > 0$, $j = 1, 2, 3$.

As δ increases, the solution λ_2 increases, λ_3 decreases, and at $\delta = \delta_{\mathrm{cr}}$ a bifurcation of the solutions of (5.1) occurs: λ_2 and λ_3 coalesce, after which they become complex conjugate (Fig. 4). For parameters (3.6), $\delta_{\mathrm{cr}} \approx 0.89$. There are no other bifurcations of λ or ω, because $\beta(\lambda_j) \neq 0, \infty$. Thus, the only possible bifurcation of the positions of the branch points is the coalescence of λ_2 and λ_3. After the coalescence, for definiteness, we will assign index "2" to the root with $\mathrm{Im}\,\lambda_2 > 0$.

The topographic positioning of branch points after the bifurcation is shown in Fig. 5b. At the branch point corresponding to λ_2, the branches k_4 and $k_{\mathrm{bl}-}$ coalesce, and at the two points corresponding to λ_3, k_3 coalesces with k_4 and k_2 with $k_{\mathrm{bl}+}$. As before, these points have no relation to instability.

For any $\delta \geq 0$, the solution λ_1 remains real and negative, while $\beta(\lambda_1)$ is real and positive. No bifurcations of the corresponding branch points occur. As a result, in the presence of a boundary layer, the branch point ω^* responsible for instability will always have a positive imaginary part.

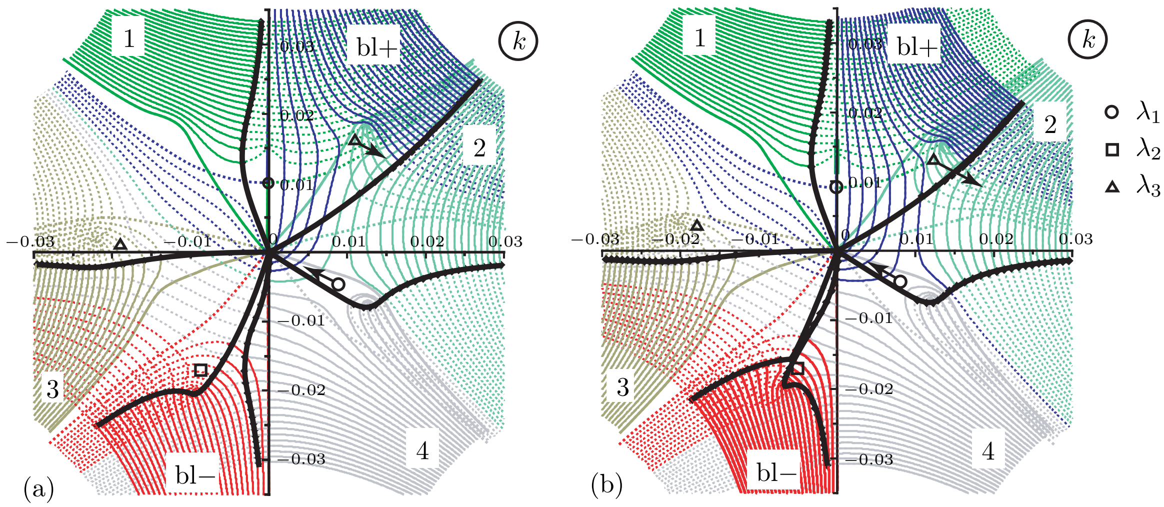

Fig. 6. Level lines $\mathrm{Re}\,\omega(k) = \mathrm{const} \geq 0$ in the complex plane k for parameters (3.6) and $M_\mathrm{w} = 0$: (a) before "switching," $\delta = 2.2$, and (b) after "switching," $\delta = 3.0$. Notation is the same as in Fig. 5.

Let us prove that $\mathrm{Im}\,\omega^*$ decreases monotonically as δ increases, i.e.,

$$(\lambda\beta^2(\lambda))'_\delta > 0$$

for $\lambda = \lambda_1$. It suffices to consider the case of $D = a = b = 1$; the general case can be reduced to it by a linear transformation of λ and δ. First, we calculate

$$\frac{d\lambda}{d\delta} = -\frac{\partial G/\partial\delta}{\partial G/\partial\lambda} = -\frac{5\lambda^3 + \lambda}{15\delta\lambda^2 - 6\lambda + \delta}.$$

Next, using the definition of $\beta(\lambda)$, we find

$$(\lambda\beta^2(\lambda))'_\delta = \frac{3 - 5\delta\lambda}{(1 - \delta\lambda)^2}\left[\lambda'(3 - 5\delta\lambda) + \frac{2\lambda}{1 - \delta\lambda}(\delta\lambda)'(1 - 5\delta\lambda)\right].$$

For $\lambda = \lambda_1 < 0$, the factor in front of the square brackets is positive. The first term in the brackets is positive. Moreover,

$$(\delta\lambda)'_\delta = \lambda + \delta\lambda' = \frac{\lambda^2(10\delta\lambda - 6)}{15\delta\lambda^2 - 6\lambda + \delta} < 0;$$

i.e., the second term in the brackets is also positive. The assertion is proved.

Thus, in the absence of tension, the boundary layer weakens the instability (reduces $\mathrm{Im}\,\omega^*$ monotonically) but cannot completely suppress it ($\mathrm{Im}\,\omega^* > 0$).

Under further increase in δ, the branches k_2 and k_4 coalesce at the branch point ω^* only up to a certain δ, after which the branches $k_{\mathrm{bl}+}$ and k_4 coalesce at this point (Fig. 6). This "switching" from k_2 to $k_{\mathrm{bl}+}$ occurs without bifurcation of the branch points themselves. For the parameters (3.6), this occurs for $\delta \approx 2.55$. For greater δ, other types of "switching" occur; however, their analysis is beyond the scope of the present study. It is only important that branches from different groups always coalesce at this point; hence, this point still leads to instability, although this instability is determined by other waves. Thus, instability in the form inherent in a uniform gas flow does not take place when the thickness of the boundary layer exceeds the indicated value: now absolute instability is due to the interaction between the wave of the boundary layer that travels along the flow and the wave of the plate that travels against the flow. Note that such an interaction, but in an incompressible fluid and with viscosity taken into account, was studied in [4].

6. STRETCHED PLATE AND THIN BOUNDARY LAYER

In the absence of a boundary layer ($\delta = 0$), the instability remains absolute for $M_\mathrm{w} < M_\mathrm{w}^\mathrm{cr}$. For $M_\mathrm{w} = M_\mathrm{w}^\mathrm{cr}$, two branch points coalesce, after which the point corresponding to instability has real frequency. In the presence of a boundary layer, system (3.4), (3.5) is too complex to be treated analytically; however, we can determine the effect of the boundary layer on stability for small δ. To this end, we linearize (3.4), (3.5) with respect to δ:

$$\begin{aligned} 4Dk^3 + 2M_\mathrm{w}^2 k - \frac{\mu}{ia}\left(1 + 3\frac{\delta b k^2}{ia\omega}\right) &= 0, \\ 2M_\mathrm{w}^2 k^2 - 4\omega^2 - \frac{3\mu k}{ia}\left(1 + \frac{1}{3}\frac{\delta b k^2}{ia\omega}\right) &= 0. \end{aligned} \tag{6.1}$$

Let k_0 and ω_0 be solutions to this system for $\delta = 0$. For small δ, we set $k = k_0 + \delta\widetilde{k}$ and $\omega = \omega_0 + \delta\widetilde{\omega}$, substitute this into (6.1), and linearize. We obtain

$$12Dk_0^2\widetilde{k} + 2M_\mathrm{w}^2\widetilde{k} + 3\mu\frac{b}{a^2}\frac{k_0^2}{\omega_0} = 0,$$

$$4M_\mathrm{w}^2 k_0\widetilde{k} - 8\omega_0\widetilde{\omega} - \frac{3\mu\widetilde{k}}{ia} + \mu\frac{b}{a^2}\frac{k_0^3}{\omega_0} = 0;$$

hence

$$\widetilde{k} = -3\mu\frac{b}{a^2}\frac{k_0^2}{\omega_0}\frac{1}{12Dk_0^2 + 2M_\mathrm{w}^2}, \qquad \widetilde{\omega} = \frac{\mu}{8\omega_0}\frac{b}{a^2}\frac{k_0^2}{\omega_0}\left(-3\frac{4M_\mathrm{w}^2 k_0 + 3i\mu a^{-1}}{12Dk_0^2 + 2M_\mathrm{w}^2} + k_0\right). \tag{6.2}$$

Now, the question of finding the sign of $\operatorname{Im}\widetilde{\omega}$ reduces to the study of the properties of branch points for $\delta = 0$.

Of principal interest is the case of $M_\mathrm{w} > M_\mathrm{w}^\mathrm{cr}$, when $\operatorname{Im}\omega_0 = 0$ and the stability is determined only by the sign of $\operatorname{Im}\widetilde{\omega}$. Notice that the denominator of the fraction in the formula for $\widetilde{\omega}$ is the derivative of the left-hand side of the first equation in (4.1). For $M_\mathrm{w} = M_\mathrm{w}^\mathrm{cr}$, when the roots of the first equation in (4.1) coalesce, this denominator vanishes. We can easily prove that for $M_\mathrm{w} > M_\mathrm{w}^\mathrm{cr}$ this denominator is positive at $k_0 = k_0^*$. Now, consider the numerator of the same fraction. This numerator also vanishes when the roots coalesce, and for $M_\mathrm{w} > M_\mathrm{w}^\mathrm{cr}$ it is purely imaginary with positive imaginary part at $k_0 = k_0^*$. Since $\operatorname{Im} k_0^* < 0$, the expression in parentheses has a negative imaginary part; hence, $\operatorname{Im}\widetilde{\omega}^* > 0$.

Thus, for $M_\mathrm{w} > M_\mathrm{w}^\mathrm{cr}$, a thin boundary layer leads to a shift of the branch point ω^* from the real axis to the upper half-plane; i.e., it gives rise to the absolute instability of an infinite plate.

In the case of a plate of finite length, this branch point also gives rise to instability; however, for $M_\mathrm{w} > M_\mathrm{w}^\mathrm{cr}$, the branches k_2 and k_3, rather than k_2 and k_4, coalesce at this point. This means that the type of this flutter is single-mode. We find that although there is no coupled-mode flutter, the left endpoint of the asymptotic curve Ω_1 [6], which corresponds to single-mode flutter, appears in the upper half-plane owing to the effect of the boundary layer. It is interesting that the appearance of this endpoint in the upper half-plane does not depend on the profile of the boundary layer, while the position of Ω_1 for $|\omega| \gg \mu^{2/3}$ can be shifted, under the effect of the boundary layer, either upward or downward, depending on the profile of the boundary layer [16].

7. STRETCHED PLATE AND BOUNDARY LAYER OF FINITE THICKNESS

In the case of finite δ, system (3.4), (3.5) was analyzed numerically. Figure 7 shows the results of calculations: the positions of the branch points for the parameters (3.6) and $0 \le M_\mathrm{w} \le 0.25$ (for these parameters, $M_\mathrm{w}^\mathrm{cr} \approx 0.13$) for $\delta = 0, 0.5, 1.0, 1.5, 2.0$. One can see that the point ω^* remains in

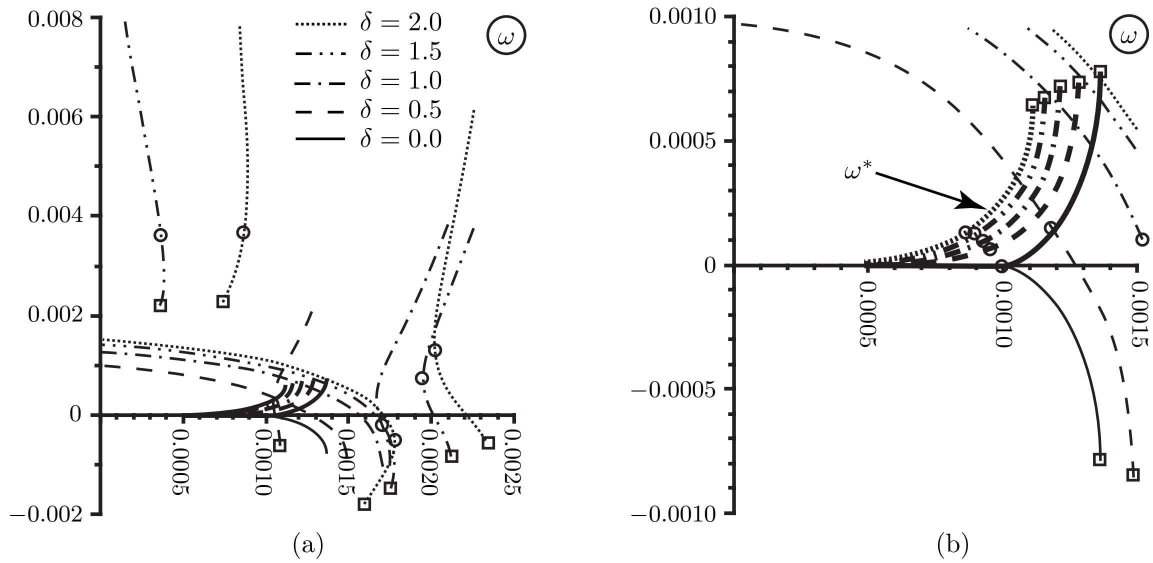

Fig. 7. (a) Motion of the branch points in the plane ω for parameters (3.6) and $0 \le M_\mathrm{w} \le 0.25$, $\delta = 0, 0.5, 1.0, 1.5, 2.0$; (b) expanded domain of small ω. Heavy lines show the frequency ω^*. Squares indicate the values for $M_\mathrm{w} = 0$, and circles, for $M_\mathrm{w} = M_\mathrm{w}^\mathrm{cr} \approx 0.13$.

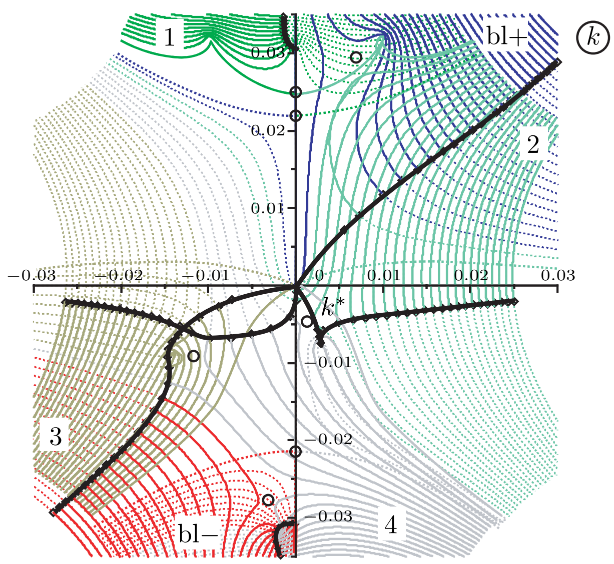

Fig. 8. Level lines $\operatorname{Re}\omega(k) = \mathrm{const} \ge 0$ in the complex plane k for the parameter values (3.6), $M_\mathrm{w} = 0.15$, and $\delta = 1.5$. Notation is the same as in Fig. 5.

the upper half-plane for all indicated δ. The topographic positioning in Fig. 8 shows that this is a unique branch point at which the roots from different groups coalesce. There are no bifurcations of branch points in this range of parameters.

Thus, the instability caused by the boundary layer persists even for finite δ from the range of parameters under consideration.

8. CONCLUSIONS

We have investigated coupled-mode flutter of a plate of large width in a gas flow with a boundary layer and also studied the character of the instability of an infinite plate. In the absence of tension and for increasing thickness of the layer, the plate remains unstable, but the growth rate of perturbations decreases. For a certain thickness, the character of the waves responsible for instability changes: in a thin boundary layer, just as in a uniform flow, instability arises due to the interaction of two waves generated by the plate; but in a sufficiently thick layer, a wave generated by the plate interacts with a wave generated by the boundary layer.

Under sufficiently large tension, when the plate is stabilized in a uniform flow, the boundary layer leads to the destabilization of the plate irrespective of the velocity and temperature profile of the flow.

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