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EXISTENCE AND UNIQUENESS OF STEADY STATE OF ELASTIC TUBES CONVEYING POWER LAW FLUID

A.B. Poroshina, V.V. Vedeneev

Institute of Mechanics, Lomonosov Moscow State University, 1 Michurinsky Prospect, 119192, Moscow, Russia, e-mail: poroshina@imec.msu.ru, vasily@vedeneev.ru

Abstract. Instability of elastic tubes with flowing fluid has been studied in many papers theoretically and experimentally in the context of biological applications. Up to the present day, only Newtonian fluid flowing in collapsible tubes has been studied. However, there are circumstances when blood, gall, and other biological fluids have essentially non-Newtonian properties. An important feature of the biological fluids motions in elastic vessels is the possibility of an oscillatory flow. Several mechanisms are possible for the appearance of such flow conditions. When the pressure inside of the tube is substantially lower than external pressure, the tube loses stability that yields the appearance of oscillations. Oscillations can also start due to non-existence of the steady state of the tube conveying fluid. In this paper, existence and uniqueness of axisymmetric states of collapsible tube conveying power law fluids have been theoretically studied. The qualitative analytical investigation of the equations of the motion has been conducted for the stationary state of the tube. As a result, the stationary state for the motion of an inviscid fluid with a given velocity profile always exists for short tubes and for arbitrarily long tubes under certain conditions. However, when viscosity is taken into account, the tube can only have a finite length, which leads to the appearance of non-stationary motion, as the only possible. Moreover, the state of tube satisfying the boundary conditions can be non-unique depending on the Reynolds number and certain additional conditions.

Key words: one-dimensional models, stationary state, ununiqueness, elastic tubes, power law fluids, occillation.

INTRODUCTION

It is known that changes in the geometry of blood vessels, bile ducts and other biological vessels, such as compressions or kinks, can lead to qualitative changes in the flow of a biological fluid, yielding disfunctions of the human or animal bodies. This kind of tube oscillation affects the flow limitation and the pressure drop. In this connection, a great interest had been shown to the oscillation of elastic tubes and their biological application [16, 18, 27, 28].

The simplest theoretical one-dimensional models were first used to analyze the stability of elastic tubes in [16, 34]. Each tube is characterized by the correspondence of the cross-sectional area and the transmural pressure (the difference between internal and external pressures). Initially, when the transmural pressure is negative and the flow velocity is large, the tube loses stability, collapses, and its resistance increases. In turn, this leads to the decrease of the flow rate and the increase of the transmural pressure. Then, the tube puffs out, the flow speed increases and the transmural pressure again drops, closing the cycle of oscillations. During each oscillation cycle, the deformation of the tube is not axisymmetric. Besides, the flow inside is quite complex, the separation of the flow from the surface of the tube exists during the part of the oscillation period.

The stationary flow of an inviscid fluid in an elastic tube was investigated in [32] based on a one-dimensional model. It was proved that there are regions in the parameter space where such a flow does not exist, and if it exists, then in the general case it is not unique. The same problem was solved in [15] with the flow energy loss due to separation from the wall taken into account at the exit from the constricted segment. It turned out that solution always exists in such formulation of the problem, although the non-uniqueness of the solution remains in the general The results of the paper have qualitative agreement case. a with the experiment, but the authors of [15] pay attention to the importance of including viscosity of the fluid in the model not only in the separation region but throughout of the flow.

Limitations of existing one-dimensional models and their improvements based on two-dimensional models are considered in [28]. A significant development of one-dimensional models and their qualitative agreement with the three-dimensional model is demonstrated in [40]. A lot of papers are devoted to analytical and numerical calculations of fluid flow in the elastic tube based on a two-dimensional model [21, 23, 24].

Despite this, the modelling of blood flow has been developed using one-dimensional models in the biomechanics during the last twenty years [1–3, 8, 9, 11, 14, 17, 30, 31, 36, 38, 39], and also simultaneous using of one-dimensional and three-dimensional models [6, 7]. The authors of [1, 2] consider the calculation technique and structure of software for numerical modelling of the hemodynamics of the cardiovascular system and the study of blood circulation in the body as a whole under the influence of periodic heartbeat. One-dimensional models of blood flow and pressure wave propagation in the arterial system are considered in [11], which are alternatives to more complex three-dimensional models. These models are used to study the effect of geometric and mechanical arterial modification on the pulsed wave, for example, due to a stenosis or a prosthesis. A review can be found of one-dimensional models for modelling the network of blood vessels in [31] and also various models that quantitatively correctly describe the pulse wave propagation are given.

On the other hand, investigations of elastic tubes have been considered only for Newtonian (linearly viscous) fluids. Although, it is known that biological fluids such as blood [4, 20] or bile [20] may have non-Newtonian properties under certain conditions. The blood parameters of the cardiac cycle given in [37] are equivalent to the index of the power law fluids n=1/8 in the rheological law. Also, the influence of pseudoplastic properties on the velocity distribution in large arteries is noted in paper [13].

In all studies, where the modelling of the vascular network is based on an one-dimensional statement of a problem [31], an unsteady calculation has to start with some steady state, which is the solution of the corresponding boundary value problem for an ordinary differential equation. In the present paper, it is proved that this state is in general not unique. In addition to the regular state of the inflated vessel corresponding to positive transmural pressure, there are a number of states in which the vessel has one or more compressions. In consequences of the calculation of the unsteady problem, in general, different results can be obtained depending on the choice of the initial state, only one of which is physically correct. Thus, additional control of the solution is necessary when the calculation of hemodynamic problems is conducted, which was not previously studied in the literature.

In this paper, we consider the axisymmetric steady state of an elastic tube conveying power law fluids based on the one-dimensional model. The tube model takes into account its radial stiffness and longitudinal tension. A qualitative analytical study of the equation describing such state is conducted. In general, the method of investigation is similar to [15, 32], but as distinct from them, we take into account the viscosity of the fluid, non-Newtonian rheology of the fluid and corresponding laminar velocity profile in the tube. The conditions for the existence of a stationary state are derived. It is proved that in the general case such a state is not unique and its possible types are investigated, as in the papers [15, 32]. As an example, stationary blood flows in a human vascular system are considered.

1. FORMULATION OF THE PROBLEM

1.1. Equations of motion

Consider a cylindrical tube of length L with elastic walls and non-Newtonian fluid flowing inside (Fig. 1). The flow and the tube interact with each other through to the equality of normal stress at the wall and matching of the flow region with the shape of the tube. The longitudinal motion of the tube wall is neglected. Let us introduce cylindrical coordinate system with z axis directed along the tube, where the tube occupies the domain $z \in [0; L]$. The flow is assumed to be axisymmetric.

We assume that the tube motion is so that each cross-section S(z,t) stays circular, and the tube points move only in a radial direction. Then, the tube geometry is defined by its radius R(z,t) in the point z at time t.

Taking tube inertia and longitudinal tension of the tube wall into account, the motion of the tube wall is written as:

$$\beta(R-R_0) - N \frac{\partial^2 R}{\partial z^2} = P - m \frac{\partial^2 R}{\partial t^2}, \qquad (1)$$

where $\beta = \frac{Eh}{(1-v^2)R_0^2}$ is the radial stiffness of the tube, *m* is the wall surface density, $N = \sigma h$

is longitudinal tension, R_0 is undeformed tube radius, E and v are the Young's modulus and the Poisson's ratio, h is the tube wall thickness, σ is the normal stress, P is the tube tension stress.

The fluid rheology obeys the power law:

$$\tau^{ij} = 2\mu \left(\sqrt{2}I_2(e)\right)^{n-1} e^{ij}, \ I_2 = \sqrt{e^{ij}e_{ij}},$$
(2)

where τ^{ij} is the viscous stress tensor, e^{ij} is the strain velocity tensor.

This law (2) is a generalisation to a three-dimensional case of the pure shear relationship:

$$\tau^{12} = \mu \left(\frac{\partial v^1}{\partial x_2}\right)^n.$$

System of equations for the incompressible fluid motion free from body forces in an arbitrary coordinate system is written as:

$$\begin{cases} \operatorname{div}(\vec{v}) = 0, \\ \frac{dv^{i}}{dt} = -\frac{1}{\rho} \nabla^{i} p + \frac{1}{\rho} \nabla_{j} \tau^{ij}, \quad i = 1, 2, 3. \end{cases}$$
(3)

The no-slip condition is assigned on the surface of the tube.



Fig. 1. Perturbation of cylindrical tube

Fig. 2. Velocity profiles for various n

1.2. One-dimensional equation describing the steady state

Consider static state of a tube conveying fluid. Let, the flow rate Q = Q(z) and radius R = R(z) does not depend on t. Then, $\partial Q / \partial z = 0$ according to the mass conservation, i.e. $Q \equiv Q_0$, where Q_0 is arbitrarily specified flow rate.

Considering the parameter $v_r / v_z \sim R / L \sim \varepsilon$ to be, we will assume that the change of the radius of the tube along the axis is sufficiently slow, so that a Poiseuille velocity profile is formed in each cross-section [22] (Fig. 2):

$$v_{z}(r) = \frac{Q_{0}}{\pi R^{2}} \frac{1+3n}{1+n} \left(1 - \left(\frac{r}{R}\right)^{(n+1)/n}\right).$$
(4)

In the case of Newtonian fluid (n=1), we obtain a standard parabolic flow.

We will integrate the momentum equation across the cross-section (the second equation of the system (3)) to obtain:

$$\int_{S(z,t)} \left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) dS = -\int_{S(z,t)} \frac{1}{\rho} \frac{\partial p}{\partial z} dS + \frac{2^{\frac{n-1}{2}}}{\rho} \int_{S(z,t)} \left[\frac{\partial}{\partial r} \left(\mu I_2^{n-1} \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \right) + \frac{\mu}{r} I_2^{n-1} \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) + \frac{\partial}{\partial z} \left(2 \mu I_2^{n-1} \left(\frac{\partial v_z}{\partial z} \right) \right) \right] dS,$$

where ρ is the fluid density.

Using the velocity profile (4), we transform the obtained expression:

$$\int_{S(z,t)} v_r \frac{\partial v_z}{\partial r} dS = \int_{S(z,t)} v_z \frac{\partial v_z}{\partial z} dS = \frac{1}{2} \frac{d}{dz} \left(\frac{(3n+1)Q^2}{(2n+1)\pi R^2} \right)$$

Under the assumption of a constant cross-section pressure (which is valid for Poiseuille problem), pressure term is rewritten as follows:

$$\int_{S(z,t)} \frac{1}{\rho} \frac{\partial p}{\partial z} dS = \frac{\pi R^2}{\rho} \frac{dp}{dz}.$$

The first two viscous terms can be transformed to the form:

$$\frac{2^{\frac{n+1}{2}}\pi}{\rho}\int_{0}^{R}\left[\frac{\partial}{\partial r}\left(\mu I_{2}^{n-1}\left(\frac{\partial v_{z}}{\partial r}+\frac{\partial v_{r}}{\partial z}\right)\right)+\frac{\mu}{r}I_{2}^{n-1}\left(\frac{\partial v_{z}}{\partial r}+\frac{\partial v_{r}}{\partial z}\right)\right]dr=\frac{2^{\frac{n+1}{2}}\pi R}{\rho}\left(\mu I_{2}^{n-1}\left(\frac{\partial v_{z}}{\partial r}+\frac{\partial v_{r}}{\partial z}\right)\right).$$

After some algebra calculation with the long-wave approximation taking into account, the following expression is obtained:

$$\frac{2^{\frac{n+1}{2}}\pi R}{\rho}\mu I_2^{n-1}\left(\frac{\partial v_z}{\partial r}+\frac{\partial v_r}{\partial z}\right)=-\frac{\mu}{\rho}\frac{2(3n+1)^nQ^n}{n^n\pi^{n-1}R^{3n-1}}$$

The last viscous term

$$\frac{2^{\frac{n-1}{2}}}{\rho} \int_{S(z,t)} \frac{\partial}{\partial z} \left(2\mu I_2^{n-1} \left(\frac{\partial v_z}{\partial z} \right) \right) dS$$

has the order ε^2 and can be neglected.

Further, as a matter of convenience, the problem is considered in a dimensionless form. The flow rate Q_0 , the inlet radius of the tube section R_0 and the fluid density ρ are chosen as the dimensionally independent scales. The dimensionless parameters (denoted by a tilde) are expressed in terms of the dimensional parameters:

$$\tilde{\beta} = \frac{R_0^5}{\rho Q_0^2} \beta, \ \tilde{N} = \frac{R_0^3}{\rho Q_0^2} N, \ \tilde{P}_{in} = \frac{R_0^4}{\rho Q_0^2} P_{in}, \ \tilde{R}e = \frac{\rho R_0^n \left(\frac{Q_0}{\pi R_0^2}\right)^{2-n}}{\mu} \frac{8n^n}{(3n+1)^n},$$
(5)

where $\tilde{R}e$ is the generalized Reynolds number of Metzner-Reed [27].

Omitting tildes, as a result, the dimensionless equation is written as:

$$\frac{d}{dz}\left(\frac{(3n+1)Q^2}{(2n+1)\pi R^2}\right) + \frac{16Q^n}{\pi \operatorname{Re} R^{3n-1}} + \pi R^2 \frac{\partial P}{\partial z} = 0$$

After the substitution of the resulting ratio of the dimensionless expression for pressure (1), the latter equation is transformed to:

$$\frac{NR^{3n+1}}{\beta}\frac{d^{3}R}{dz^{3}} + \left(\frac{2(3n+1)R^{3n-4}}{(2n+1)\pi^{2}\beta} - R^{3n+1}\right)\frac{\partial R}{\partial z} - \frac{16}{\pi^{2}\beta \text{Re}} = 0.$$
 (6)

It requires three boundary conditions. Two of them are obvious and correspond to a fixed radius of the tube at its ends. For example, a tube is putted on the ends of metal tubes in experiments on Starling resistor. Also, we will assume that the transmural pressure $P = P_{in}$ at the leading end of the tube is known:

$$R(0) = R(L) = 1, \ \frac{d^2 R(0)}{dz^2} = -\frac{P_{in}}{N}.$$

Thus, the equation (6) with these boundary conditions is the dimensionless boundary value problem describing the stationary state of the tube conveying fluid. Equation (6) can be rewritten to:

$$\frac{d}{dz} \left[-\frac{N}{\beta} \frac{d^2 R}{dz^2} + \left(\frac{3n+1}{2(2n+1)\pi^2 \beta} R^{-5} + 1 \right) R_s \right] = -\frac{16}{\pi^2 \beta \text{Re} R^{3n+1}}.$$
(7)

The one-dimensional equation (7) was obtained and investigated in [42] without taking the tension into account (i.e., for N=0). In this case, it transforms into an equation of the first order, and it is integrated in an explicit form. For a given inlet radius R(0) = 1, the solution is always unique and exists only for a finite length $L < L_{max}$. When the tension is accounted ($N \neq 0$), equation (7) is an equation of the third order with variable coefficients and is not integrated in an explicit form. The following sections are devoted to a qualitative study of the properties of the boundary value problem (7).

2. SOLUTION OF THE INVISCID PROBLEM

2.1. Analysis of equilibrium points

First, we consider the case when the Reynolds number tends to infinity, $\text{Re} \rightarrow \infty$, then the differential equation (7) has the first integral:

$$-\frac{N}{\beta}\frac{d^{2}R}{dz^{2}} + \left(\frac{3n+1}{2(2n+1)\pi^{2}\beta}R^{-5} + 1\right)R + A = 0,$$
(8)

where according to the boundary conditions for z = 0

$$A = -\left(\frac{P_{in}}{\beta} + \frac{(3n+1)}{2(2n+1)\pi^2\beta} + 1\right).$$

Let us denote R = x and $\partial R / \partial z = y$. Then, taking (8) into account, we obtain the system:

$$\begin{cases} x' = y \\ y' = \frac{3n+1}{2(2n+1)\pi^2 N x^4} + \frac{\beta x}{N} + \frac{\beta A}{N}, \end{cases}$$
(9)

where the prime denotes the derivative with respect to z.

Under the new notations, the remaining boundary conditions are written as:

$$x(0) = x(L) = 1.$$
(10)

In order to find a solution of the boundary value problem (9), (10), it is necessary to choose the initial value y(0) on the line x(0) = 1, such that the value of z, for which the trajectory x(z), y(z) returns to the line x=1 is equal to the length of the tube L.

We will conduct the qualitative phase plane for the system (9) analysis. First, we find the equilibrium points (x_0, y_0) of the system (9). Obviously $y_0 = 0$ and x_0 is a solution of equation

$$T(x) = x^{5} + Ax^{4} + \kappa = 0, \tag{11}$$

$$\kappa = \frac{3n+1}{2(2n+1)\pi^2\beta} > 0.$$

It can be shown that equation (11) can have no more than three real roots. The one, always present root, is negative, and two roots are positive for $A < A_{cr}$. Moreover, one root is greater than $x_{cr} = -4A/5$, the other is less. The one multiple root $x = x_{cr}$ is positive and the other is negative for $A = A_{cr}$. There is only one negative root for $A > A_{cr}$. The critical value is denoted by index cr. The value

$$A_{cr} = -\frac{5}{4^{4/5}} \kappa^{\frac{1}{5}}$$

corresponds to the multiple root of equation (11), which is equal to the value x_{cr} , for which the derivative of the function T(x) becomes zero.

As an example, "T - x" diagram is plotted for $\beta = 0.7$, n = 0.2 and the values $A = A_{cr} = -1.00$, $A = -1.08 < A_{cr}$ ($P_{in} = -0.005$), $A = -0.37 > A_{cr}$ ($P_{in} = -0.500$) (Fig. 3).

As far as x has the sense of the tube radius, only the positive roots $x_0 > 0$ have a physical meaning. It can be shown that the trajectories of the system (9) cannot intersect the axis x=0, therefore we will restrict oneself with the study of the half-plane x > 0of the phase plane (11).

Henceforth, we need the value of the A for $P_{in} = 0$:

$$A = A_0 = -(\kappa + 1).$$

The value of A_0 coincides with the value of A_{cr} for $\kappa = 1/4$, and in other cases always $A_0 < A_{cr}$, because the function

$$A_{cr} - A_0 = -\frac{5}{4^{4/5}} \kappa^{\frac{1}{5}} + \kappa + 1$$

is always positive for $\kappa \neq 1/4$ (Fig. 4).

Let us consider the types of equilibrium points for $A \le A_{cr}$ (in the opposite case there are no equilibrium points $x_0 > 0$). After linearization, the second equation of the system (9) with respect to the point $(x_0, 0)$ becomes:

$$\begin{cases} x' = y, \\ y' = \frac{\beta}{N} \left(5 + \frac{4A}{x_0} \right) x. \end{cases}$$
(12)

The roots λ_1 and λ_2 of the characteristic equation

$$\begin{vmatrix} -\lambda & 1\\ \frac{\beta}{N} \left(5 + \frac{4A}{x_0} \right) & -\lambda \end{vmatrix} = 0$$

determine the behavior of the phase trajectories of the system (12) on the phase plane.

Depending on the eigenvalues

$$\lambda = \pm \sqrt{\frac{\beta}{N} \left(5 + \frac{4A}{x_0} \right)},$$

we obtain:

1) the first equilibrium point $x_0 > -4A/5$ is saddle;



Fig. 3. Function graph T(x)

Fig. 4. Function graph $((A_{cr} - A_0))$

2) the less equilibrium point $x_0 < -4A/5$ is center.

We consider three possible types of phase trajectories depending on A. A number of examples are given below for the parameters

$$\beta = 0.7, N = 2.4, n = 0.2,$$
 (13)

which correspond to $A_0 = -1.08$, $A_{cr} = -1.00$. Let us introduce the value of the inlet pressure P_{cr} , wherein $A = A_{cr}$:

$$P_{cr} = \beta \left(\frac{5}{4^{4/5}} \kappa^{\frac{1}{5}} - (\kappa + 1) \right) < 0.$$

1. $A_0 < A < A_{cr}$.

The inlet pressure in the tube P_{in} is negative, but not too large in absolute value $(0 > P_{in} > P_{cr})$. There are two equilibrium points, one of which is the center, and the other is the saddle.

1.1. For $\kappa < 1/4$, both points correspond to $x_0 < 1$, i.e. the narrowed state of the tube. For example, Fig.5 shows the phase plane for (13) and $P_{in} = -0.010$ (A = -1.06). As the absolute value of the transmural pressure increases, the equilibrium points approach to each other. Namely, the point corresponding to the center moves in the positive direction of the axis OX, and the point corresponding to the saddle moves from $x_0 = 1$ (one of the possible values of x_0 for $P_{in} = 0$) in the negative direction of the axis OX. An example of the phase plane for (13) and $P_{in} = -0.040$

(A = -1.02) is shown in Fig. 6.

1.2. For $\kappa > 1/4$, both points correspond to $x_0 > 1$. When $|P_{in}|$ increases, the points approach to each other similar to the previous case. An example of the phase plane is shown in Fig.7 for

$$\beta = 0.1$$
, $N = 2.4$, $n = 0.2$, $P_{in} = -0.007$.



2. $A \ge A_{cr}$.

The inlet pressure P_{in} is negative and large enough in absolute value $(0 > P_{cr} \ge P_{in})$. The equilibrium points merge for $A = A_{cr}$, and they disappear for $A > A_{cr}$. An examples of phase planes are shown in Fig. 8 for values (13) and $P_{in} = P_{cr}$ $(A = A_{cr})$ or in Fig. 9 for $P_{in} = -0.100$ $(A = -0.93 > A_{cr})$.

3. $A < A_0$, the value $P_{in} > 0$ (the inlet pressure increases).

In this case, there are two equilibrium points, one of which is the center, and the other is the saddle. Moreover, the saddle point has the value $x_0 > 1$, and the center has the value $x_0 < 1$, which corresponds to the inflated and constricted states of the tube accordingly. The phase plane for (13) and $P_{in} = 0.040$ ($A = -1.13 < A_0$) is shown in Fig. 10.

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As the transmural pressure increases, the equilibrium point corresponding to the center moves closer to 0, and the equilibrium point corresponding to the saddle moves from 1, toward the positive direction the *OX* axis. An example of the phase plane for (13) and $P_{in} = 0.500$ ($A = -1.79 < A_0$) is shown in Fig.11.

As the system (9) does not have singularities for x > 0 and, consequently, does not suffer other bifurcations of the phase plane, except the merging and disappearance of equilibrium points at $A = A_{cr}$. Thus, the examples of phase planes shown above for the cases 1, 2, 3 do not qualitatively change under other parameters of the problem.

Analysis of the maximal possible length of the tube

Below, we will investigate the value of z = L at which the phase trajectory returns on the line x = 1 for x(0) = 1 and different values of y(0).

If the initial value $y(0) \rightarrow +\infty$ then $dx/dz \rightarrow +\infty$, and from the structure of phase planes it is seen that the tube is unrestricted inflate, so that the solution of the boundary value problem does not exist.

Let us prove that if the initial value $y(0) \rightarrow -\infty$, then the trajectory travel time before its return to the line x = 1 (value of z) tends to 0. The phase plane for (13) and $y(0) \rightarrow -\infty$ can be an example of such a trajectory (Fig. 12).

The trajectory can be decomposed into three sections:

1. $y \rightarrow -\infty$.

In this section, the trajectory is close to a straight line parallel to the OX axis. According to the first equation of the system (9), x' = y, the value $dx/dz \rightarrow -\infty$, which guarantees a high speed of passage of this section. At the same time, the value of x (distance traveled) is not greater than 1, which means that the value of z (the time of passage of this section) will tend to 0.

2. The section of the trajectory from $y \rightarrow -\infty$ to $y \rightarrow +\infty$

In this section, the value of $x \rightarrow 0$ and the y rate was an order of $1/x^4$, according to the second equation of the system, and the other terms can be neglected. Throwing off them, we consider simplified equations for determining the time of motion of the trajectory along the vertical section

$$\begin{cases} x' = y, \\ y' = \frac{3n+1}{2(2n+1)\pi^2 N x^4}, \end{cases}$$

or

$$x'' = \frac{3n+1}{2(2n+1)\pi^2 N x^4}.$$
 (14)

Taking into account the relation

$$x'' = y' = \frac{\partial y}{\partial x} \frac{\partial x}{\partial z} = \frac{\partial y}{\partial x} y,$$

equation (14) can be integrated to obtain

$$y^{2} = -\frac{3n+1}{(2n+1)\pi^{2}N3x^{3}} + C,$$
(15)

where due to the initial condition x(0) = 1,

$$C = \frac{3n+1}{3(2n+1)\pi^2 N} + y(0)^2.$$

Equation (15) is integrated the second time, resulting in the solution of the equation (14):

$$\int \frac{d\zeta}{\sqrt{C - \frac{3n+1}{3(2n+1)\pi^2 N\zeta^3}}} = \pm z + C_2, \ C_2 = \text{const}.$$

Owing to the initial condition $C_2 = 0$, so that the solution becomes:



$$P_{in} = -0.050, A_0 < A \le A_{cr}$$

Fig. 14. Phase plane for $x_0 > 1$, $P_{in} = -0.007, A_0 < A \le A_{cr}$

$$\int_{x}^{1} \frac{d\zeta}{\sqrt{C - \frac{3n+1}{3(2n+1)\pi^2 N\zeta^3}}} = z$$

Solution is rewritten in the form:

$$\left(\frac{3n+1}{3(2n+1)\pi^2 N}\right)^{\frac{1}{3}} \frac{1}{C^{5/6}} \int_{\chi(C)x}^{\chi(C)} \frac{d\xi}{\sqrt{1-\xi^{-3}}} = z,$$
(16)

when changing variables

$$\zeta = \xi \left(\frac{3n+1}{3(2n+1)\pi^2 NC} \right)^{\frac{1}{3}},$$
$$\chi(C) = \left(\frac{3n+1}{3(2n+1)\pi^2 NC} \right)^{-\frac{1}{3}}.$$

The integral in (16) at $\xi \to 1$ (approximation of the vertical section to x = 0) is convergent and has an order of $\chi(C)$, i.e. an order of $C^{1/3}$, and therefore z has an order of $C^{-1/2} \sim |y(0)|^{-1}$. Thus, the time z of passing the vertical section of the trajectory tends to 0 for $y(0) \to -\infty$.

3. $y \rightarrow +\infty$.

In this case, the trajectory is close to a straight line parallel to the OX axis, and the value of z tends to 0, similar to the trajectory section for $y \rightarrow -\infty$.

Hence, $L \to 0$ for $|y(0)| \to +\infty$, therefore the maximum length of the tube L_{\max} , for which a solution of the boundary value problem exists, can be achieved only for limited |y(0)|.



Fig. 15. Phase plane for (13), $P_{in} = 0.1$, $A < A_0$

For the limited |y(0)|, the value of L can be infinitely large only if the trajectory passes near the saddle point, since in its neighborhood the speed on the trajectory falls almost to zero, and it can remain in this neighborhood arbitrary long time. If the value x_0 of the center point is 1 (i.e. $\kappa > 1/4$, $P_{in} = 0$), then the tube will not inflate and constrict, and always retain its initial shape, so the length of the tube L can be infinitely large. In other cases, the center point can not suit, because the phase trajectories are closed around this critical point in its neighborhood, hence the trajectory starting at x(0) = 1 can not approach the center point too close.

Let us now analyze the values of L_{max} for the previously examined three cases:

1. $A_0 < A \le A_{cr}$.

In this case, the tube can be arbitrarily long only for y(0) close to the separatrix corresponding to the saddle point.

1.1. An example of such a trajectory is shown for equilibrium points $x_0 < 1$ and values (13), $P_{in} = -0.050$, A = -1.01, $y(0) \approx -0.051$, close to the separatrix of the saddle point (Fig.13).

1.2. For equilibrium points $x_0 > 1$, an example is the trajectory for the values of the parameters $\beta = 0.1$, N = 2.4, n = 0.2, $P_{in} = -0.007$, $y(0) \approx 0.025$ (Fig.14).

2. $A > A_{cr}$.

Since there are no equilibrium points, the value of L_{max} is finite. Thus, steady states of the tube conveying fluids for sufficiently long tubes do not exist in this case.

3. $A < A_0$.

In the case, which corresponds to a positive inlet transmural pressure, the tube can also be arbitrarily long for y(0), sufficiently close to the separatrix corresponding to the saddle point. An example is the trajectory for the parameters (13) and $P_{in} = 0.100$, A = -1.01, $y(0) \approx 0.085$, corresponding to the separatrix of the saddle point (Fig. 15).



Fig. 18. Phase plane for (17), $P_{in} = -0.003, A_0 < A \le A_{cr}, L = 25$



Consequently, the tube can be arbitrarily long for a steady state only for $A \le A_{cr}$ (which corresponds to $P_{in} \ge P_{cr}$).

Non-uniqueness of the solution

In this section, we consider the non-uniqueness of the solution of the boundary value problem (9), (10) depending on the parameter A for different tube lengths.

Most of examples discussed below are given for the dimensionless parameters

$$\beta = 0.4, \quad N = 0.2, \quad n = 0.2, \tag{17}$$

then $A_0 = -1.14$, $A_{cr} = -1.12$.

1. $A_0 < A \le A_{cr}$.

1.1. In this case, for equilibrium points $x_0 < 1$ and for any values of the length L, there is a non-uniqueness of the solution, namely, there are two trajectories along which it is

possible to return to the line x=1 for the same value of L. One solution corresponds to where is the intersection of the separatrix coming $y(0) < y_{c} < 0$, y_s from the saddle point with the line *x* = 1. solution corresponds The second to $y(0) < y_{s} < 0$, and the trajectory goes around the center point.

For example, two trajectories corresponding to L=5 are obtained for the parameters (17) and $P_{in} = -0.003$, A = -1.13 and the values $y(0) \approx -0.11400$ and $y(0) \approx -0.01635$ (Fig. 16). The tube radius R(z) corresponding to these solutions is plotted in Fig. 17. Another example, corresponding to L=25, is plotted in Figs. 18, 19 ($y(0) \approx -0.01672$ and $y(0) \approx -0.01674$). Let us consider the tube length as a function of the various initial values of y(0). The graph has the asymptote $y(0) = y_s$, corresponding to the value, where y(0) belongs to the separatrix of the saddle point. For the previously considered example, the graph has the asymptote $y(0) = y_s \approx -0.01673$ (Fig. 20) with the parameters (17). The solution exists only for negative values of y(0). Calculations show that the decrease of L is monotonically as y(0) decreases. Therefore, for any values of L there are two solutions, namely with and without passing around the center point.

1.2. If the equilibrium points correspond to $x_0 > 1$, then the number of solutions depends on the length of the tube. For sufficiently small $L < L_{cr1}$, there are two trajectories corresponding to different initial y(0), similar to the case $A > A_{cr}$, which will be considered below. There is no solution for $L_{cr2} > L > L_{cr1}$. Next, for $L_{cr3} > L > L_{cr2}$, two more solutions appear, corresponding to one additional pass around the center point, at that they are symmetric with respect to the middle of the tube. For $L > L_{cr3} > L_{cr2}$ two additional solutions appear. For sufficiently large L, there exists any prescribed even number of solutions. Each pass around the center point corresponds to a local contraction of the tube and subsequent widening to the original radius. For example, four solutions corresponds to the value L = 40: one without pass around the center point, two with one pass, and one with two passes (Fig. 21).





The plot of L versus the initial value y(0) has two asymptotes, y_{s+} and y_{s-} , corresponding to two separatrices of the saddle point (Fig. 22). There is no solution of the boundary value problem for $y(0) > y_{s+}$ or $L_{cr1} < L < L_{cr2}$. There are two solutions for $y(0) < y_{s+}$ and a small tube length $L < L_{cr1}$. There is uniqueness of the solution for y(0) < 0 and $L = L_{cr1}$, or for $0 < y(0) < y_{s+}$ and $L = L_{cr2}$. There is no solution of the boundary value problem or $y(0) < y_{s-}$ and $L > L_{cr1}$. There may exist a number of solutions corresponding to the number of returns to the line x=1 in the case of passing around the center point for $y_{s-} < y(0) < y_{s+}$, and for sufficiently large tube lengths $(L > L_{cr2})$.



 $P_{in} = 0.003, \ y(0) \approx -0.025, \ y(0) \approx 0.016$

Fig. 27. Function $A_0 < A \le A_{cr}$ for (17), $P_{ir} = 0.003, y(0) \approx -0.025, y(0) \approx 0.016$

2. $A > A_{cr}$.

Taking into account that there are no equilibrium points, the solution does not exist for large values of *L*. In other words, there is no trajectory returning to the point x(L) = 1. However, there are two solutions for sufficiently small values of *L*, that is there are two different trajectories corresponding to different initial values y(0) and the same values *L*, such that x(L) = 1.

Since L=0 for y(0)=0 and $y(0) \rightarrow -\infty$, the plot of the tube length L against the initial value y(0) (Fig. 23) has a maximum corresponding to the uniqueness of the solution. Thus, there is no solution for $L > L_{max}$, there is a unique solution for $L = L_{max}$, there are two solutions for $L < L_{max}$. The value y(0) is negative, because the existence of a solution is possible only in this case.

For example, the phase plane is plotted for the parameters (17) and $P_{in} = -0.100$, A = -0.89 in Fig. 24. There are two different trajectories corresponding to ythe same value L = 0.83 for different initial values $y(0) \approx -0.2$, $y(0) \approx -2$. Both solutions are plotted in Fig. 25 as a function of the tube radius *R* against the *z* coordinate. The function *L* versus y(0) has a maximum $L_{cr} \approx 1.955$ at the point $y(0) \approx -0.52$ (Fig. 23). 3. $A < A_0$.

In this case, when the saddle point $x_0 > 1$ and the center $x_0 < 1$, the solution is always a non-uniqueness. There are two trajectories corresponding to different initial y(0) for sufficiently small values of L. Similarly to the $A_0 < A \le A_{cr}$ case, one solution corresponds to $0 < y(0) < y_{s+}$, and the other solution where the trajectory passes around the center point corresponds to $y(0) < y_{s-} < 0$.

For example, two trajectories corresponding to L=7.5 exist for $P_{in} = 0.003$ (A = -1.15) and the initial values $y(0) \approx -0.025$, $y(0) \approx 0.016$ (Fig. 26, 27).

For $L > L_{cr1} > 0$, two more solutions appear, corresponding to one additional pass around the center point, and the first turns into the second when reflected from the middle of the tube. Four additional solutions appear for $L > L_{cr2} > L_{cr1}$. There exists any prescribed number of solutions for sufficiently large L. Each pass around the center point corresponds to a local contraction of the tube and subsequent widening to the original radius.

For example, eight solutions correspond to the value L = 20, namely, one without a pass around the center point, three with one pass, and four with two passes (Fig. 28).

The plot of L versus the initial value y(0) has two asymptotes, y_{s+} and y_{s-} , corresponding to the two separatrices of the saddle point (Fig. 29). There is no solution of the boundary value problem $y(0) > y_{s+}$. There for are two solutions $y(0) < y_{s+}$ and sufficiently small L. There is no solution for for $y(0) < y_{s-}$ and the large value L. For $y_{s-} < y(0) < y_{s+}$ and for large values of the tube length L, there exist a number of solutions corresponding to the number of returns to the line x=1 for the case of passes around the center point.

4. SOLUTION OF THE VISCOUS PROBLEM

If the Reynolds number is finite, then in equation (8) the variable $A \neq \text{const}$, namely A = A(z) is an increasing function of z. Moreover, the derivative of this function is bounded from below by a constant D(A'(z) > D > 0) for limited values R(z) < M(M = const). Therefore, the motion of the trajectory x(z), y(z) can be represented as motion along the vector field (9), with simultaneous evolution of this field due to the growth of A. Since A = A(z) is growing function, then the problem reduces to the case $A > A_{cr}$ for some finite z. Physically, this corresponds to a drop of pressure below P_{cr} due to viscous losses. After that, the trajectory leaves the domain $x \le 1$ at a finite time and does not return to the line x = 1. Hence, the solution of the boundary value problem exists only for $L \le L_{max}(\text{Re})$, and if the length L_{max} is exceeded, there is no steady state.

There can be a non-uniqueness of the solution, similar to the inviscid case, depending on Re and, accordingly, the rate of A increase. For example, two trajectories corresponding to





Fig. 29. Function L(y(0)) for (17), $P_{in} = 0.003$, and shapes of $A_0 < A \le A_{cr}$ curves corresponding to each solution L = 10 are obtained for (17), $P_{in} = 0$ and Re = 1000 (Fig. 30). At that, the tube constriction is located closer to the tube end in the solution with pass around center point.

The function R(z) is plotted for (17), $P_{in} = -0.003$, Re = 250 and $y(0) \approx -0.092$, $y(0) \approx -0.018$ (Fig. 31, solid lines) for comparison with the results obtained for the inviscid problem in Fig. 17 (see Fig. 31, dashed lines). It is seen that the constriction of the tube is shifted under the influence of viscosity towards the tube end in both solutions. In addition, the constriction of the tube becomes smaller in a solution with pass around center point, and the constriction of the tube increases in a solution without pass around center point.

A nonuniqueness is possible for large finite Reynolds number Re and positive input transmural pressure $P_{in} > 0$ (and also for $P_{cr} < P_{in} < 0$, $\kappa > 1/4$), similar to the case $A < A_0$ with several passes around center point. As an example, Fig. 32 shows the plot R(z) for the parameters (17), $P_{in} = 0.03$ and Re = 1500. Solution with two constrictions corresponds to y(0) = 0.0745, y(0) = -0.0316 and y(0) = 0.0292, solution with one constriction corresponds to y(0) = -0.0317 and y(0) = 0.0293, the widening exists for y(0) = -0.0746 (Fig. 32).

5. BLOOD VESSELS

The human cardiovascular system consists of the heart and blood vessels, such as arteries, capillaries and veins. Blood moves along the vessels under the influence of pressure drop, namely from high to low pressure region. Considering that the vascular wall is elastic, changes in transmural pressure are accompanied by changes in diameter and tension.

The mechanical properties and geometric parameters of blood vessels are presented in Table and previously were studied in a number of papers. The dimensionless parameters (denoted as by the tilde) are expressed in terms of the dimensional parameters by (5) and $\tilde{L} = L/R_0$. The radial stiffness can be found as $\beta = \frac{Eh}{(1-v^2)R_0^2}$, as far as it is known that the range of values of the Young's module *E* for blood vessels is 0.1–1.0 MPa [5]

the range of values of the Young's module *E* for blood vessels is 0.1-1.0 MPa [5] and the average value is usually about 0.4 MPa [41], and the value of the Poisson's ratio $v \approx 0.4$.



Fig. 30. Function $A_0 < A \le A_{cr}$ for (17), $P_{in} = 0$, Re = 1000





Fig. 32. Function $A_0 < A \le A_{cr}$ for (17), $P_{-} = 0.03 \text{ Re} = 1500$

Fig. 33. Function $A_0 < A \le A_{cr}$ for ascending aorta

Table

	R_0 , cm [43]	<i>h</i> , cm [12]	V ₀ , cm/s [43]	Re [43]	P_{in} , N/cm ² [32]
Ascending aorta	1.00 - 1.60	0.20	22.00-63.00	1600-5800	1.33
Large arteries	1.00 - 3.00	0.10	20.00 - 50.00	110 - 850	1.10
Large veins	0.25 - 0.50	0.05	15.00 - 20.00	210 - 570	0.13
Vena cava	0.10	0.15	11.00 - 16.00	630 - 900	0.10

Parameters of blood vessels

In its turn, the longitudinal tension can be found as N = Ehl, where l is the longitudinal strain of the vessel. In calculations, the tension $(E \times l)$ equals to 5 N/cm² [29], which corresponds to the stresses that the wall of the vessel experiences in the human body. The index of the power law fluids is assumed equal to one (n = 1) in the calculations, because the blood flow in large vessels usually corresponds to Newtonian properties.

According to the calculations, solutions exist for parameters corresponding to the aorta and cava, and it is non-unique on all occasions. Two trajectories corresponding to the real length of the vessel L = 4 cm [25] are obtained for the values of the ascending aorta parameters ($R_0 = 1.50 \text{ cm}$ [25], $V_0 = 30 \text{ cm/s}$, Re = 1900) (Fig. 33).

Another example is an artery. Their walls are thick enough to stand the high pressure that is created under powerful blood ejections. Also, walls do not fall off, if they are not filled with blood. Two trajectories exist for the parameters of the arteries (for example, Table 1 left renal artery $R_0 = 0.25 \text{ cm} [25]$, $V_0 = 35 \text{ cm/s}$, Re = 370, L = 3.00 cm [25] or right common iliac artery $R_0 = 0.44 \text{ cm} [25]$, $V_0 = 35 \text{ cm/s}$, Re = 650, L = 5.80 cm [25]), one of which corresponds to slight inflation, and the second corresponds to a significant constriction at the tube end (Fig. 34, Fig. 35).



Fig. 34. Function $A_0 < A \le A_{cr}$ for left renal artery

Fig. 35. Function $A_0 < A \le A_{cr}$ for right common iliac artery



Fig. 36. Function $A_0 < A \le A_{cr}$ for left internal Fig. 37. Function $A_0 < A \le A_{cr}$ for precava iliac vein

In its turn, the pressure in the veins is lower then the pressure in the arteries, so the walls of the veins are thinner. When considering the correspondence of the radius against the tube length with the parameters corresponding to the veins (for example, left internal iliac vein $R_0 = 0.25$ cm [25], $V_0 = 20$, Re = 210, L = 3.00 cm [25] or precava $R_0 = 1.10$ cm [25], $V_0 = 15$ cm/s, Re = 700, L = 8.00 cm [25]), two trajectories exist, similar to the trajectories obtained for the arteries. (Fig. 36, Fig. 37).

6. DISCUSSION

When a viscous fluid flows in an elastic vessel, a pressure drop occurs due to viscous losses, which leads to a gradual decrease in the diameter of the vessel due to the elasticity of the walls. In its turn, the decrease of the radius causes higher pressure drop than would be in a vessel of constant cross-section, which in turn produces a more intense contraction of the cross-section. It was shown in [42] that the length of vessel without tension can only

be finite, namely, the radius decrease becomes infinitely fast for $L = L_{max}$, and the solution of the problem does not exist for $L > L_{max}$.

In this paper, it is proved that the tension cannot essentially change this result and prevent the collapse of the vessel as its length increases. In other words, the steady flow of the power low fluid in a tensioned elastic vessel is possible only for a limited length of the vessel $L < L_{max}$, because of the viscosity of the fluid causing the pressure losses. In long vessels, the flow can only be unsteady, which may result in the phenomenon of flutter in elastic tubes conveying fluid [16, 29].

This situation differs fundamentally from the results of [15], where it is shown that if the viscosity is neglected, but the separation of the flow from the wall in the contraction is taking into account, the solution exists for arbitrarily long tubes.

Also in the present paper, it is proved that when the solution of the boundary value problem of an elastic vessel conveying power law fluid exists $(L < L_{max})$, it is always nonunique for sufficiently large Reynolds numbers.

For negative transmural pressure, in addition to the "natural" solution corresponding to a gradual radius decrease and subsequent increase due to tension (in the terms of this study, this solution is without pass around center point), there is a second solution in which the radius decrease is more significant (the solution with pass around center point), Fig. 31. The second solution has a greater predisposition to both the flow separation from walls and a loss of axisymmetry of the vessel, considering a more drastic and stronger contraction of the cross section. Apparently, this solution cannot be realized in real vessels.

For positive transmural pressure, in addition to the "natural" solution, in which the vessel is inflated, there is a solution family in which there is one or more additional constrictions (solutions with passes around center point), Fig. 32, which, apparently, cannot be realized in the vascular system for the same reason.

As a result, just one of the few existing solutions corresponds to the "natural" state of the blood vessel. Meanwhile, any of these solutions can be realized in the numerical solution of the boundary value problem, since they are all equivalently correct in the context of the one-dimensional model. Since at the present day one-dimensional models of the network of blood vessels are one of the main instruments for the study of hemodynamic problems [1-3, 8, 9, 11, 14, 17, 30, 31, 36, 38, 39], the fact of non-uniqueness of steady flow is extremely important. Calculations of the dynamics of the pulse wave in a network of vessels always begin with some initial condition, which is natural to choose to be a steady flow. If this initial condition is not chosen correctly, then the subsequent dynamics will not be calculated correctly. Thus, the results of this paper show that additional control of the physical correctness of obtained solutions is necessary, because of the non-uniqueness of the steady state.

CONCLUSIONS

In the paper, we derived a one-dimensional equation of the steady state of an elastic tube conveying power law fluid, which takes the fluid rheology and the resulting laminar velocity profile into account.

For the motion of an ideal fluid (the Reynolds number $\text{Re} \rightarrow \infty$) with a given velocity profile, the conditions for the existence of a steady state for arbitrarily long tube length are obtained, namely, the value of the inlet transmural pressure should be either positive, or negative and greater than some critical value ($P_{in} > 0$ or $P_{cr} < P_{in} < 0$). When viscosity is taken into account (finite Reynolds numbers), the tube can always have only a finite length.

oscillations can be associated with the previously known oscillatory states, caused by the drop of pressure in the fluid, loss of stability of the tube and its periodic collapses.

For sufficiently large Reynolds numbers and $L < L_{max}$, there is a non-uniqueness of the steady state of the tube that corresponds to the boundary conditions of the problem. Namely, two solutions exist for $P_{in} < P_{cr} < 0$ or $P_{cr} < P_{in} < 0$ and $\kappa < 1/4$, and a multiple-solution family exists for $P_{in} > 0$ or $P_{cr} < P_{in} < 0$ and $\kappa > 1/4$. One of the solutions corresponds to a slight contraction or inflation of the tube with a smooth radius change along the length. The second solution (or solution family) corresponds to steep local tube contractions with subsequent extension to the original radius. In real life, the existence of such states is apparently impossible in connection with the possible loss of the tube axisymmetry and the separation of the flow in the local constriction. The model considered here does not describe these phenomena. It is likely that the reorganization to such a regime actually leads to the appearance of an oscillatory nature of the flow.

The results show that additional control of the states obtained in the numerical calculations in hemodynamics based on one-dimensional models is necessary in connection with the revealed non-uniqueness of the solution. Namely, only one of possible solutions of the boundary value problem is physically correct (in the terms of this paper, this solution is without pass around the center point), the other have one or more nonphysical vessel compressions (corresponding to the pass around the center point). The absence of such control in numerical modelling can lead to incorrect results of calculations, which do not reflect the real motion of the biofluid in the vessel.

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