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Influence of the boundary layer on flutter of elastic plate in supersonic gas flow

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ABSTRACT: We investigate the stability of an infinite elastic plate in a supersonic gas flow. This problem has been studied in many papers regarding panel flutter problem, where uniform flow is usually considered. In this paper, we take the boundary layer on the plate into account and investigate its influence on plate stability. Unsteady pressure is derived from Rayleigh equation, which governs inviscid disturbances of the boundary layer. If the wave length is not too large, Rayleigh equation is solved analytically; otherwise it is studied numerically. In both cases dispersion relation for a plate in supersonic flow with boundary layer is derived, and the influence of the boundary layer on the plate stability is analysed.

KEY WORDS: panel flutter; boundary layer; aeroelasticity.

1 INTRODUCTION

The classical stability theory of shear flows deals with fluid flows over rigid surfaces. Many different ways of laminar-turbulent transition control have been studied, such as cooling or heating of the surface, boundary layer suction or blowing, surface porosity, etc.

Following [1], a series of papers is devoted to the investigation of boundary layer stability over compliant surfaces [2–9]. It has been shown that elastic and viscous properties of the surface can significantly change the shape of the neutral stability curve and can change instability character from convective to absolute. Also, in addition to inviscid inflection-point instability and viscous Tollmien-Schlichting instability, a series of new instability types appears due to flexibility of the surface [2, 3].

The stability of compressible shear flows, and especially of supersonic flows over a compliant wall, is studied much less. In supersonic flows over a compliant wall, one more instability type appears, namely, panel flutter [10, 11]. It is dangerous not because of flow transition to turbulence, but primarily because of high-amplitude vibrations of the wall structure. This phenomenon is well known in aviation and has been studied in numerous papers since the 1950s.

Up to recent years, only one panel flutter type, namely, coupled-mode flutter, was studied. It occurs due to the coupling of two plate eigenmodes through gas flow. In case of low supersonic flow speeds, another flutter type exists, namely, single-mode flutter. Even though this type of flutter was discovered in the 1960s [11], it is still viewed by some as being non-physical and only appearing in calculations due to insufficient accuracy in numerical studies. However, recently, it was analytically proved [12] that this type of flutter indeed exists; later, it was observed in experiments [13] and studied numerically [14, 15].

The overwhelming majority of panel flutter investigators did not take the boundary layer into account and consider uniform velocity and temperature profile. In a few papers where the influence of the boundary layer was numerically studied [16–18], a particular velocity profile was considered, namely, $(1/7)^{th}$ power velocity law. The reason is that those studies were devoted to the modelling of experiments [19, 20]. The same boundary layer profile was studied in [21]. Those studies showed that the boundary layer of this particular profile can decrease the growth rate of unstable eigenmodes or even fully suppress instability. However, in flows over flight vehicles at different flight conditions and for different skin panel locations, boundary layer profiles over panels can significantly differ. In this paper, we study the influence of the arbitrary boundary layer profile on panel flutter.

2 FORMULATION OF THE PROBLEM

We investigate the stability of an infinite elastic plate stretched by an isotropic in-plane force. One side of the plate is exposed to shear gas flow; the other side experiences constant pressure equal to an undisturbed pressure of the flow so that the undisturbed state of the plate is flat (Figure 1). The flow has a boundary layer over the plate surface with given velocity and temperature profiles $u_0(z)$ and $T_0(z)$, respectively. It is assumed that they are governed by the flow conditions over the flight vehicle, whose single panel is represented by the plate considered.

The problem is investigated in 2-D formulation; also, we neglect the growth of the boundary layer so that the unperturbed flow does not depend on x. This admits the perturbations of a travelling-wave type, where all variables are proportional to $e^{i(kx-\omega t)}$. All variables are assumed to be non-dimensional, with the speed of sound and temperature of the flow outside the boundary layer taken as the velocity and temperature scales, the plate thickness as the length scale, and plate material density as the density scale.

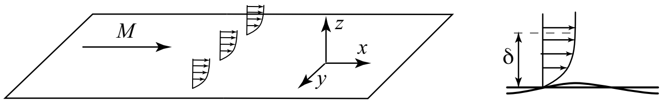

Figure 1: Gas flow over elastic plate.

The plate is governed by Kirchhoff-Love small deflection plate theory. In a dimensionless form, the plate equation is as follows:

$$D\frac{\partial^4 w}{\partial x^4} - M_w^2\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial t^2} + p(x,0,t) = 0, \qquad (1)$$

where $w(x,t)$ is the plate deflection, D is the dimensionless plate stiffness, M_w is the square root of the dimensionless in-plane tension force, and $p(x,z,t)$ is the flow pressure disturbance generated by the plate, which hence is a function of w.

The flow is assumed to be laminar; the Reynolds number $\mathrm{Re}\to\infty$. This means that small perturbations of the flow are governed by the inviscid Rayleigh equation, while viscosity is essential only in the formation of the steady boundary layer as non-uniform distribution of velocity and temperature.

Let us now proceed to equations. Let $v(z)e^{i(kx-\omega t)}$ and $p(z)e^{i(kx-\omega t)}$ be the perturbations of vertical velocity component and pressure. Then the compressible Rayleigh equation [22] takes the following form:

$$\frac{d}{dz}\left(\frac{(u_0-c)dv/dz - v du_0/dz}{T_0-(u_0-c)^2}\right) - \frac{1}{T_0}k^2(u_0-c)v = 0. \quad (2)$$

Amplitude of pressure perturbation is expressed through $v(z)$ as follows:

$$p(z) = \frac{\mu}{ik}\frac{(u_0-c)dv/dz - v du_0/dz}{T_0-(u_0-c)^2}, \qquad (3)$$

where μ is the dimensionless density of the flow outside the boundary layer.

Consider boundary conditions for the Rayleigh equation. First, at the plate surface $z=0$, we assign the condition of impenetrability along the plate bent in the shape of a travelling wave: $w(x)=e^{i(kx-\omega t)}$. The second boundary condition is the radiation condition as $z\to+\infty$. We apply it at the outer boundary of the boundary layer $z=\delta$, where δ is the dimensionless boundary layer thickness, as follows. Assume that for $z>\delta$, the flow is uniform, $u_0\equiv M_\infty$ is the Mach number outside the boundary layer, and $T_0\equiv 1$. Then for $z>\delta$, Rayleigh equation (2) is reduced to an equation with constant coefficients, whose solution is $v(z)=Ce^{\gamma z}$, $\gamma=-\sqrt{k^2-(M_\infty k-\omega)^2}$. The radiation condition as $z\to+\infty$ yields a particular square root branch, namely,

$$\mathrm{Re}\,\gamma < 0 \qquad (4)$$

for $\mathrm{Im}\,\omega \gg 1$. This exponential solution outside the boundary layer must be matched with the solution inside the boundary layer. Thus, the boundary conditions finally take the following form:

$$v=-i\omega \quad (z=0), \qquad \frac{1}{v}\frac{dv}{dz}=\gamma \quad (z=\delta). \qquad (5)$$

Hereafter, we assume that the parameter μ (which is the density of the flow outside the boundary layer rated to the plate material density) is small, and use asymptotic expansions as $\mu\to 0$ wherever it is possible. In industrial applications, the order of μ is typically in the range $10^{-5}\dots 10^{-3}$.

3 DISPERSION RELATION FOR AN INFINITE PLATE IN A GAS FLOW

We assume that the plate is infinite and consider travelling-wave disturbances: $w(x,t)=e^{i(kx-\omega t)}$. In this section we derive the dispersion relation that connects wave number k and frequency ω in the following manner. First, we solve Rayleigh equation (2) with boundary conditions (5). Second, we calculate disturbance of pressure (3) acting on the plate surface. Finally, we substitute the pressure disturbance into the plate equation (1) and obtain the dispersion relation.

Note that the Rayleigh equation can have two singularities [22]. The first one is the critical point z_c, where $u_0(z_c)=c$. It leads to the singularity of the solution that will be discussed below. The other one is the point where $T_0(z)-(u_0(z)-c)^2=0$, which means that the phase speed of the wave is equal to the local speed of sound. This singularity is removable.

3.1 Long waves

The Rayleigh equation is solved by using two methods, analytical for long waves and numerical for short waves. First, consider the case of long waves. The solution can be constructed in the form of a convergent series in k^2, known as the Heisenberg expansion [22, 23]. In industrial applications related to flutter, wavelengths $\lambda=2\pi/k$ of practical interest are usually much larger than the boundary layer thickness δ. Therefore, we can assume that $|k|$ is small, namely, $|k|\ll 1/\delta$, and keep only the first term of the series. This is equivalent to neglecting the second term of order of k^2 in (2):

$$\frac{d}{dz}\left(\frac{(u_0-c)dv/dz - v du_0/dz}{T_0-(u_0-c)^2}\right) = 0.$$

This equation is easily solved, and its general solution is

$$v(z) = \left(c_1\left(\int_0^z \frac{T_0(\zeta)d\zeta}{(u_0(\zeta)-c)^2} - z\right) + c_2\right)(u_0(z)-c). \quad (6)$$

Substitution into (3) yields the pressure perturbation

$$p(z) \equiv \frac{c_1\mu}{ik}. \qquad (7)$$

It is clearly seen that (6) has a singularity in the critical point that is not removable in inviscid theory. Solutions that are limits of the viscous system solutions (analogous to the Orr-Sommerfeld equation in incompressible fluid) as viscosity vanishes are constructed such that the integration path is located in the compex z plane and passes below the critical point [22]. In particular, if $\mathrm{Im}\,c>0$ (growing disturbances), then integration can be accomplished along the real z axis. If $\mathrm{Im}\,c\leqslant 0$ (neutral and damped disturbances), integration must be accomplished in the complex z below the singularity.

Substituting (6) into boundary conditions (5), finding c_1 and c_2, and calculating (7) yield pressure disturbance on the plate surface

$$p(0) = -\mu \left(\left(\frac{(M_\infty k - \omega)^2}{\sqrt{k^2 - (M_\infty k - \omega)^2}} \right)^{-1} + \left(\int_0^\delta \frac{T_0(\zeta)d\zeta}{(u_0(\zeta) - c)^2} - \delta \right) \right)^{-1}.$$

The first term in parentheses represents the contribution of the uniform flow outside the boundary layer, whereas the second term represents the contribution of the boundary layer.

It is convenient to extract boundary layer thickness δ from the integral. Substituting $\zeta = \delta\eta$, we obtain the following:

$$\int_0^\delta \frac{T_0(\zeta)d\zeta}{(u_0(\zeta) - c)^2} = \delta \int_0^1 \frac{T_0(\eta)d\eta}{(u_0(\eta) - c)^2}.$$

Hereafter, we will consider functions $u_0(\eta)$ and $T_0(\eta)$ as describing the boundary layer profile, where η is the local boundary layer vertical coordinate, $0 \le \eta \le 1$.

3.2 Short waves

For the case of short wave length comparable with the boundary layer thickness, the Rayleigh equation is solved numerically through the Runge-Kutta method. Having velocity and temperature boundary layer profiles, $u_0(z)$ and $T_0(z)$, respectively, we first find the critical point z_c and choose a smooth path passing below the critical point in the complex z-plane (figure 2). The next step is the numerical solution of the Rayleigh equation along that path, by using the shooting method and the Runge-Kutta method. Finally, by using formula (3), we calculate unsteady pressure $p(0)$ on the plate surface.

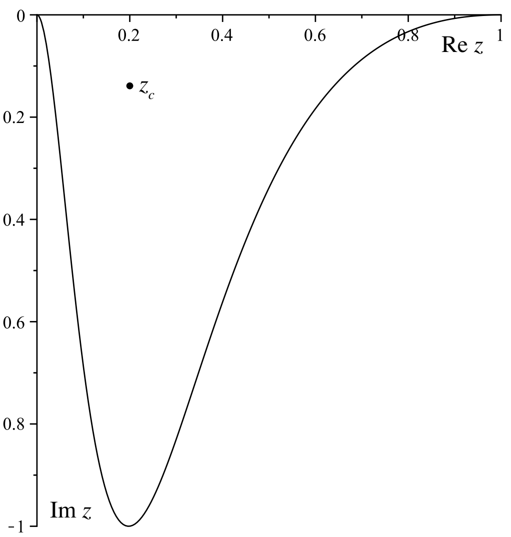

Figure 2: Integration path chosen for solving the Rayleigh equation for the velocity profile $u_0(\eta) = M_\infty \sin(\pi\eta/2)$, $M_\infty = 1.6$, $c = 0.5 - 0.33i$. The critical point is $z_c \approx 0.20 - 0.14i$.

3.3 Dispersion relation

The substitution of plate deflection $w(x,t) = e^{i(kx-\omega t)}$ and pressure disturbance $p = p(0)e^{i(kx-\omega t)}$ into the plate equation (1) yields the dispersion relation. In case of long waves it takes the form

$$\mathcal{D}(k,\omega) = (Dk^4 + M_w^2 k^2 - \omega^2) - \mu \left(\left(\frac{(M_\infty k - \omega)^2}{\sqrt{k^2 - (M_\infty k - \omega)^2}} \right)^{-1} + \delta \left(\int_0^1 \frac{T_0(\eta)d\eta}{(u_0(\eta) - c)^2} - 1 \right) \right)^{-1} = 0. \quad (8)$$

Note that its structure reflects the contribution of each of three media: the plate, the boundary layer, and the uniform flow outside the boundary layer. In particular, the expression in the first parentheses represents the plate: its three terms reflect the bending stiffness, tension, and inertia of the plate. The expression in the second parentheses represents the flow, whose influence is proportional to μ. The first term in the parentheses is the contribution of the uniform flow, whereas the second represents the boundary layer.

As $\delta \to 0$, the dispersion relation (8) coincides with the dispersion relation for a plate in uniform flow [12, 24]:

$$(Dk^4 + M_w^2 k^2 - \omega^2) - \mu \frac{(M_\infty k - \omega)^2}{\sqrt{k^2 - (M_\infty k - \omega)^2}} = 0.$$

In the case of short waves, when unsteady pressure on the plate surface is calculated numerically, the dispersion relation can be written in the form

$$\mathcal{D}(k,\omega) = Dk^4 + M_w^2 k^2 - \omega^2 + p(0) = 0, \quad (9)$$

where $p(0)$ is found by integrating the Rayleigh equation and applying (3).

4 INSTABILITY OF TRAVELLING WAVES OF MODERATE WAVE LENGTHS

Hereunder we consider wavelengths that are not too long so that solutions $\omega(k)$ of the dispersion relation for the plate in the flow are close to those for the plate in vacuum. In particular, we assume that $|k| \gg \mu^{1/3}$. The influence of the flow on such waves is small and can be taken into account in the first approximation in μ. Investigation of long waves was conducted in [25].

In this section we consider "moderate" wave lengths, when, on one hand, the wave is not too long, and, on the other hand, not too short so that the Rayleigh equation can be solved analytically as shown in Section 3.1 (i.e., $\mu^{1/3} \ll |k| \ll 1/\delta$). Under this assumption, the Taylor expansion in μ yields the following:

$$\omega(k,\mu) = \omega(k,0) - \mu \frac{\partial \mathcal{D}}{\partial \mu} \Big/ \frac{\partial \mathcal{D}}{\partial \omega} \Big|_{\mu=0} + o(\mu).$$

Neglecting infinitesimal terms, we obtain

$$\omega(k,\mu) = \omega(k,0) - \\ - \frac{\mu}{2\omega(k,0)}\left(\left(\frac{(M_\infty k-\omega)^2}{\sqrt{k^2-(M_\infty k-\omega)^2}}\right)^{-1} + \\ \delta\left(\int_0^1 \frac{T_0(\eta)d\eta}{(u_0(\eta)-c)^2} - 1\right)\right)^{-1}, \quad (10)$$

where the expression in parentheses is calculated at $\mu = 0$.

It is seen that the frequency $\omega(k,\mu)$ of the plate in the flow can be represented as the frequency of the plate in vacuum $\omega(k,0)$, slightly (by the order of μ) modified by the flow. As the frequency in vacuum $\omega(k,0)$ is always real, the stability of the plate in the flow is governed by this small term caused by the flow. The flow influence on the frequency, as well as on the dispersion relation, is clearly split into two different mechanisms expressed by two terms in the parentheses: uniform flow (the first term) and the boundary layer (the second term).

In order to investigate the influence of the boundary layer on the growth of travelling waves, let us first consider waves in uniform flow, $\delta = 0$ [12]. In this case, the behaviour of the wave is governed by the square root in the right-hand side of (10). The choice of an appropriate branch is not obvious and must be conducted by considering radiation condition (4) for rapidly growing waves. In other words, the square root branch must be an analytical continuation of the branch defined as follows:

$$\operatorname{Re}\sqrt{k^2-(M_\infty k-\omega)^2} > 0, \quad \operatorname{Im}\omega \to +\infty,$$

from the region of very large $\operatorname{Im}\omega$ to the values of interest. We will be primarily interested in unstable solutions, that is, $\operatorname{Im}\omega > 0$; therefore, the path for continuation in ω plane can be chosen so that it lies in the upper half plane, and this continuation is single-valued (note that both branch points of the square root are real for $k \in \mathbb{R}$).

The accurate treatment of the continuation [12] yields four cases. If the wave propagates upstream (i.e. $c < 0$), it is always damped. If the wave propagates downstream, it is amplified if $0 < c < M_\infty - 1$, neutral if $M_\infty - 1 < c < M_\infty + 1$, and damped if $c > M_\infty + 1$. Physically, these inequalities express a relationship between the phase speed c of the wave running in the plate and the speed of acoustic waves $M_\infty \pm 1$ in the gas flow.

Thus, three types of wave behaviour are possible:

- In the uniform flow, the wave is growing and supersonic relating to the flow: $0 < c < M_\infty - 1$.
- In the uniform flow, the wave is neutral and subsonic relating to the flow: $M_\infty - 1 < c < M_\infty + 1$.
- In the uniform flow, the wave is damped and supersonic relating to the flow: $c > M_\infty + 1$ or $c < 0$.

Let us treat them in series in order to investigate the influence of the boundary layer term in (10).

4.1 Influence of the boundary layer on the growing wave

First, we will investigate waves of the first type. To make the analysis clearer, denote

$$A = \frac{\sqrt{k^2-(M_\infty k-\omega)^2}}{(M_\infty k-\omega)^2} = \frac{\sqrt{1-(M_\infty-c)^2}}{k(M_\infty-c)^2},$$
$$B = \delta\left(\int_0^1 \frac{T_0(\eta)d\eta}{(u_0(\eta)-c)^2} - 1\right),$$

and rewrite (10):

$$\operatorname{Im}\omega(k,\mu) = -\frac{\mu}{2\omega(k,0)}\operatorname{Im}(A+B)^{-1}.$$

As the wave is growing at $\delta = 0$, then $\operatorname{Im}A = a > 0$, $\operatorname{Re}A = 0$. Let us determine which regions of B plane correspond to a decrease or increase of $\operatorname{Im}(A+B)^{-1}$ in comparison with $\operatorname{Im}A^{-1}$ (i.e. stabilisation or destabilisation of the wave by the boundary layer).

It is easy to prove that the level lines of $\operatorname{Im}(A+B)^{-1}$ on the complex B plane are circles with centres on the imaginary axis that pass through the point $B = -ia$. If the second intersection of the circle and imaginary axis lies above this point, then $\operatorname{Im}(A+B)^{-1} < 0$; otherwise, $\operatorname{Im}(A+B)^{-1} > 0$. Level line $\operatorname{Im}(A+B)^{-1} = \operatorname{Im}A^{-1}$ is a circle that passes through the point $B = 0$. Level line $\operatorname{Im}(A+B)^{-1} = 0$ (neutral disturbances) is a horizontal line passing through the point $B = -ia$ (figure 3).

Fix the phase speed c and consider $\operatorname{Im}(A+B)^{-1}$ as a function of the boundary layer thickness δ, assuming that profiles $u_0(\eta)$ and $T_0(\eta)$ are specified. A does not depend on δ, while B is a linear function of δ. Then values of B on the complex plane that correspond to different values of δ lie on a ray that begins at $B = 0$.

Two cases are possible. If $\operatorname{Im}B \geqslant 0$, then the ray is directed upward or horizontally. In this case, for any δ, the wave is growing; its growth rate is less than at $\delta = 0$ and tends to zero as $\delta \to \infty$.

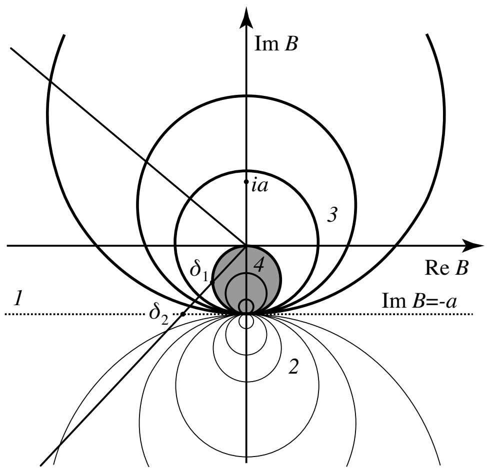

Figure 3: Level lines $\operatorname{Im}(A+B)^{-1}$. 1: $\operatorname{Im}(A+B)^{-1} = 0$; 2: set of lines below 1, $\operatorname{Im}(A+B)^{-1} > 0$; 3: set of lines above 1, $\operatorname{Im}A^{-1} < \operatorname{Im}(A+B)^{-1} < 0$; 4 (shaded): region $\operatorname{Im}(A+B)^{-1} < \operatorname{Im}A^{-1} < 0$.

In the other case, $\operatorname{Im} B < 0$, the ray is directed downward. For $0 < \delta < \delta_1$, the growth rate is positive and larger than at $\delta = 0$, where δ_1 is the value where the ray crosses the circle $\operatorname{Im}(A + B)^{-1} = \operatorname{Im} A^{-1}$ (figure 3). For $\delta_1 < \delta < \delta_2$, the growth rate is still positive, but less than at $\delta = 0$, where δ_2 is the value at which the ray crosses the line $\operatorname{Im}(A + B)^{-1} = 0$. Finally, for $\delta > \delta_2$, the wave is damped.

The value of $\operatorname{Im} B$ is calculated explicitly as follows. Integrand in the definition of B has a singularity at the critical point $\eta = \eta_c$, which must be passed below, according to Lin's rule. Let us expand boundary layer profiles in the Taylor series near $\eta = \eta_c$:

$$T_0(\xi) = T_{00}+T_{01}\xi+\dots, \quad u_0(\xi) = c+u_{01}\xi+u_{02}\xi^2/2+\dots,$$

where $\xi = \eta - \eta_c$, T_{0n} and u_{0n} are n-th derivatives in the critical point. Then we have the following:

$$\frac{T_0(\xi)}{(u_0(\xi)-c)^2} = \frac{T_{00}+T_{01}\xi+\dots}{(u_{01}\xi+u_{02}\xi^2/2+\dots)^2} = \frac{T_{00}}{u_{01}^2}\frac{1}{\xi^2} + \frac{1}{u_{01}^2}\left(T_{01} - T_{00}\frac{u_{02}}{u_{01}}\right)\frac{1}{\xi} + \text{reg.terms} \quad (11)$$

As the $1/\xi$ term is the only source of non-zero imaginary part of B, we obtain

$$\operatorname{Im} B = \frac{\pi\delta}{u_{01}^2}\left(T_{01} - T_{00}\frac{u_{02}}{u_{01}}\right) = -\pi\delta\frac{T_0^2}{u_0'^3}\left(\frac{u_0'}{T_0}\right)',$$

where the prime denotes the derivative with respect to z at the critical point.

Thus, $\operatorname{Im} B$ is a function of the boundary layer thickness and the local behaviour of the velocity and temperature profiles in a neighbourhood of the critical point. On the contrary, the value of $\operatorname{Re} B$ depends on all regular terms in the expansion (11), i.e. on the profiles in the full segment $\eta \in [0; 1]$.

Let us now reformulate results obtained in terms of the boundary layer profile. If the profile is generalised convex, i.e. $(u_0'/T_0)' < 0$ everywhere, then for any phase speed $\operatorname{Im} B > 0$. This means that the boundary layer has a stabilising effect: although the growth rate in the flow with the boundary layer stays positive, it is less than in the uniform flow.

If the profile has a generalised inflection point, then there exists a range of phase speeds c such that the effect of the boundary layer is destabilising (i.e. growth rate of the wave in the flow with the boundary layer is higher than in the uniform flow) for $\delta < \delta_1$, since $\operatorname{Im} B < 0$. The closer $|\operatorname{Re} B|$ is to zero (that is, direction of the ray to vertical), the more growth rate we obtain due to the boundary layer. In the limit case $\operatorname{Re} B = 0$, $\operatorname{Im} B \to -a$ the value of $\operatorname{Im}(A + B)^{-1}$ (and hence the growth rate $\operatorname{Im} \omega$) tends to infinity as $\delta \to \delta_1$.

The possibility of large growth rates caused by the boundary layer in real flows will be considered below. But the limit case of $c \to 0$ can be immediately excluded, as the behaviour of $\operatorname{Re} B$ can be explicitly found. After the integration of (11), the leading term of $\operatorname{Re} B$ is as follows:

$$\int_{-\eta_c}^{1-\eta_c} \frac{T_{00}}{u_{01}^2}\frac{1}{\xi^2}d\xi = -\frac{T_{00}(\eta_c)}{u_{01}^2(\eta_c)}\left(\frac{1}{1-\eta_c} + \frac{1}{\eta_c}\right).$$

Thus, as $c \to 0$ (consequently, $\eta_c \to 0$), $\operatorname{Re} B \to -\infty$, i.e. the ray direction tends to horizontal. This means that, first, $\delta_1 \to 0$ and, second, the increase of the growth rate caused by the boundary layer for $0 < \delta < \delta_1$ is negligible. We conclude that destabilisation of the wave growing in uniform flow by the boundary layer can occur only for intermediate phase speeds, $0 < c < M_\infty - 1$.

4.2 *Example: Profiles of accelerating and decelerating flows*

Consider self-similar boundary layer profiles [26] that are governed by parameter β. If $\beta > 0$, then the free-stream flow is accelerating; if $\beta < 0$, then it is decelerating; if $\beta = 0$, then the flow velocity is constant. These profiles are analogous to those boundary layer profiles that appear in incompressible fluid in case of degree function flow $u_\infty(x) = Cx^m$, $\beta = 2m/(m+1)$. We will assume that the wave length is much less than the characteristic distance, over which the flow outside the boundary layer essentially changes; hence, the flow can be considered as locally uniform. In other words, saying 'accelerating' and 'decelerating' flow hereunder, we mean the corresponding boundary layer profile, still considering the free-stream flow uniform.

In this section, we consider an example of the Prandtl number $Pr = 1$ and the heat-insulated plate so that the boundary layer profiles are expressed through the solution $f(\xi)$ of the following equation [26]:

$$f''' + ff'' = \beta(f'^2 - 1), \quad f(0) = f'(0) = 0, \quad f'(+\infty) = 1.$$

Velocity is then given as a function of a similarity variable ξ: $u_0(\xi) = f'(\xi)$. The physical z coordinate is calculated for each ξ as follows:

$$z = C\int_0^\xi T_0(u_0(\xi))d\xi.$$

Temperature profile $T_0(u_0)$ is given using the same expression as in an adiabatic flow:

$$T_0(u_0) = 1 + \frac{\gamma - 1}{2}(M_\infty^2 - u_0^2). \quad (12)$$

Let us consider several profiles as examples. They are denoted as 1–5 in figure 4a and correspond to $\beta = 2$, 0.5, 0, -0.14, and -0.199. Note that profile 5 is a limit case of the attached boundary layer, because $du_0(0)/dz < 0$ for $\beta < -0.199$, and the boundary layer separates from the plate.

Calculations have been conducted for parameters that correspond to a steel plate at 3 km above sea level:

$$D = 23.9, \quad M_w = 0, \quad \mu = 0.00012, \quad \gamma = 1.4, \quad (13)$$

and Mach number $M_\infty = 1.6$.

The generalised curvature $(u_0'/T_0)'$ is plotted for these profiles in figure 4b, while values of $\operatorname{Re} B$ and $\operatorname{Im} B$ are shown in figures 4c and 4d.

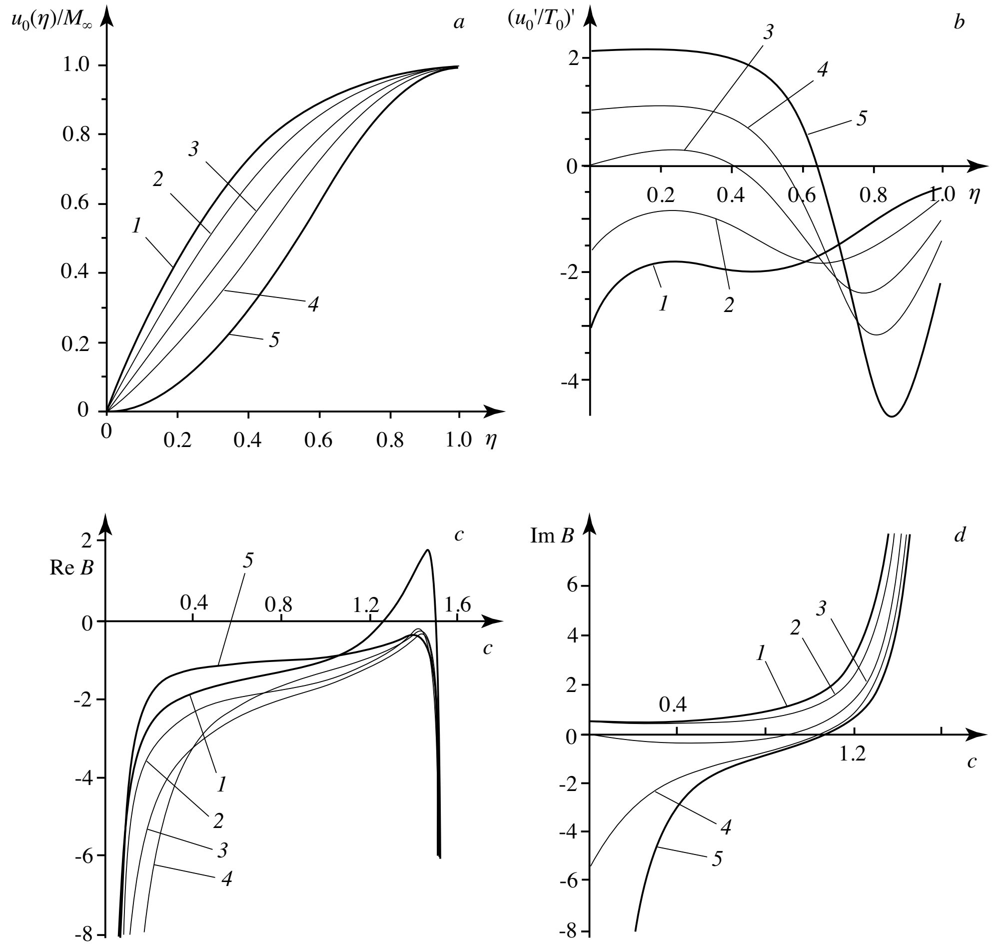

Figure 4: Velocity profiles for accelerating and decelerating flows over heat-insulated plate for parameters (13), $M_\infty = 1.6$, $Pr = 1$: (a) velocity profiles, (b) generalised curvature, (c) $\operatorname{Re} B$, (d) $\operatorname{Im} B$. Curves 1–5 correspond to $\beta = 2.0, 0.5, 0.0, -0.14, -0.199$.

It is seen in figure 4d that in the case of profiles 1 and 2, $\operatorname{Im} B > 0$, and the growth rate decreases when δ increases. This is also seen in figure 5, where $\operatorname{Im}\omega$ is plotted versus the boundary layer thickness for a particular wave number $k = 0.06$.

In the case of profiles 3–5, $\operatorname{Im} B < 0$ for waves travelling with supersonic speed relative to the outer flow (i.e. for $c < M - 1$), which means that the wave is amplified by the boundary layer at $\delta < \delta_1$ (figure 5). As proved above, the smallness of $|\operatorname{Re} B|$ is a condition of the significant destabilising effect of the boundary layer. It is seen in figure 4c that amongst the profiles considered, $|\operatorname{Re} B|$ is the smallest for profile 5 on a whole interval $c < M_\infty - 1$. According to this, $\operatorname{Im}\omega$ is significantly increased for this profile by the boundary layer (figure 5).

4.3 *Influence of the boundary layer on neutral and damped waves*

We have investigated the influence of the boundary layer on waves that are growing in potential flow, that is, $0 < c < M_\infty - 1$.

Now suppose that the wave is damped in potential flow, that is, $M_\infty + 1 < c$ or $c < 0$. Then A is purely imaginary, and $\operatorname{Im} A < 0$. There is no critical point in the flow; therefore, $\operatorname{Im} B = 0$. Then $0 < \operatorname{Im}(A + B)^{-1} < \operatorname{Im} A^{-1}$, which means that the boundary layer decreases the wave damping rate but cannot result in its growth. Damped waves stay damped.

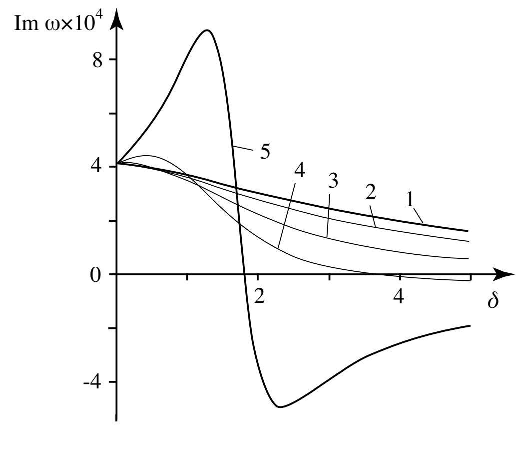

Figure 5: Function $\operatorname{Im}\omega(\delta)$ for profiles 1–5 in figure 4 for parameters (13), $M_\infty = 1.6$, $k = 0.06$.

Next, suppose that the wave is neutral in uniform flow, that is, $M_\infty - 1 < c < M_\infty + 1$, and $\operatorname{Im} A = 0$. If $M_\infty \leqslant c \leqslant M_\infty + 1$, then there is no critical point in the flow; therefore, $\operatorname{Im} B = 0$. The wave stays neutral in the boundary layer.

If the wave in uniform flow is neutral and $M_\infty - 1 < c < M_\infty$, then there is a critical point, and $\operatorname{Im} B \neq 0$. Consider two types of the boundary layer profile. In case of the flow with a generalised convex profile, $(u_0'/T_0)' < 0$, $\operatorname{sign} \operatorname{Im} B = -\operatorname{sign}((u_0'/T_0)') > 0$. Consequently, $\operatorname{Im}(A+B)^{-1} < 0$ and $\operatorname{Im} \omega > 0$, which means that the wave is destabilised by the boundary layer. Note that the boundary layer itself (i.e. over a rigid wall) is stable. This conclusion is similar to that obtained by [5], who showed that for the Blasius boundary layer in incompressible fluid, elasticity of the plate yields destabilisation of the layer as $\mathrm{Re} \to \infty$.

Finally, suppose that the boundary layer profile has a generalised inflection point z_{infl} in the subsonic part of the layer so that $(u_0'/T_0)' > 0$ for $z < z_{\text{infl}}$ and $(u_0'/T_0)' < 0$ otherwise (this is typical for boundary layers over a flat heat-insulated plate; we restrict ourselves to this class of boundary layers). Then waves that have critical points at $z > z_{\text{infl}}$, that is, $M_\infty - 1 < u_0(z_{\text{infl}}) < c < M_\infty$, are destabilised by the boundary layer because $\operatorname{sign} \operatorname{Im} B = -\operatorname{sign}((u_0'/T_0)') > 0$ and hence $\operatorname{Im}(A+B)^{-1} < 0$. On the contrary, waves with such phase speeds that $M_\infty - 1 < c < u_0(z_{\text{infl}})$ become damped.

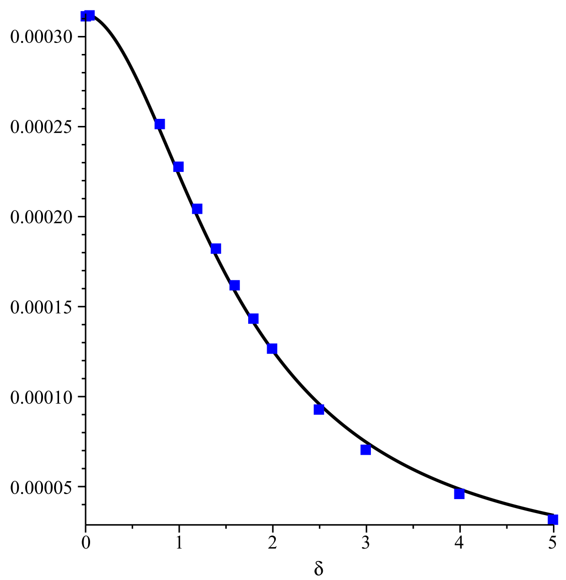

Figure 6: $\operatorname{Im} \omega(\delta)$ for $k = 0.1$. Numerical data are shown by points, analytical solution (10) is represented by the curve.

5 INSTABILITY OF SHORT WAVES

5.1 Numerical method

In this section we consider waves whose lengths are short enough so that the Rayleigh equation is solved numerically as described in Section 3.2. For solving the dispersion equation (9), we use the following procedure. Having the problem parameters, M_∞, μ, D, M_w, δ, the velocity and temperature profiles, and the wave number k, we search for a solution of the dispersion equation $\omega(k)$ using iterative method. In the first step of iterations ω_1 is taken equal to the frequency of the plate in vacuum $\sqrt{Dk^4 + M_w^2 k^2}$. Now, let us have n-th approximation, ω_n. We numerically solve the Rayleigh equation, find the velocity perturbation v, and calculate the unsteady pressure $p(0, \omega_n)$ according to (3). For the next approximation we put $\omega_{n+1} = \sqrt{Dk^4 + M_w^2 k^2 + p(0, \omega_n)}$. The procedure is repeated until the desired accuracy is achieved, i.e., $|\mathcal{D}(k, \omega_n)| < \varepsilon$.

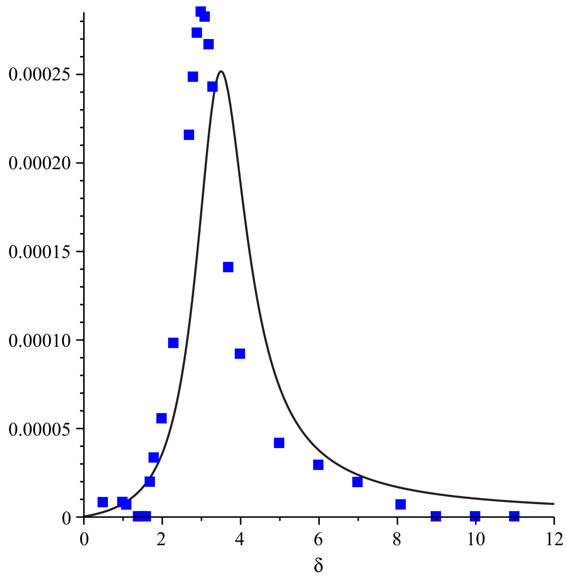

Figure 7: $\operatorname{Im} \omega(\delta)$ for $k = 0.175$. Numerical data are shown by points, analytical solution (10) is represented by the curve.

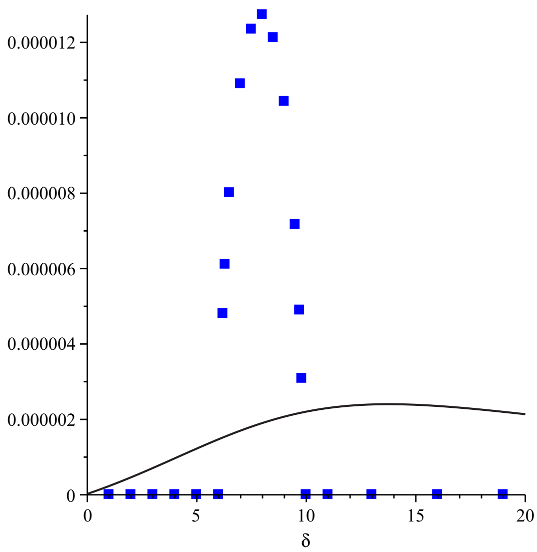

Figure 8: $\operatorname{Im} \omega(\delta)$ for $k = 0.275$. Numerical data are shown by points, analytical solution (10) is represented by the curve.

5.2 Results

We studied the velocity profile $u_0(\eta) = M_\infty \sin(\pi\eta/2)$ and the temperature profile (12) for parameters (13), $M_\infty = 1.6$. Calculated growth rate versus the boundary layer thickness, $\operatorname{Im} \omega(\delta)$, are plotted for a range of wave numbers k. For "moderate" wave lengths, $k = 0.025..0.100$, there is a satisfactory agreement between the numerical and analytical solution (10) (figure 6).

With increasing k, starting from $k = 0.125$, a difference between the numerical and analytical solution appears (figure 7) due to the second term in (2) neglected in the analytical solution. First, the maximum value of the growth rate for the numerical solution becomes higher than for analytical for sufficiently large k. Namely, numerical results give: $\max \operatorname{Im} \omega(\delta) \approx 0.0012164$, 0.0007332, 0.0000173, whereas from (10) $\max \operatorname{Im} \omega(\delta) \approx 0.0044722$, 0.0007936, 0.0000091 for $k = 0.125$, 0.15, 0.25, respectively. Second, there is some difference in the boundary layer thickness at which the maximum growth rate is achieved: numerical solution gives smaller δ comparing with analytical (figure 7, 8). The third effect of short waves is that $\operatorname{Im} \omega = 0$ for δ lying outside a certain segment, as demonstrated in figure 8, whereas (10) gives non-zero growth rates for any δ. The higher k, the more pronounced these effects are.

6 CONCLUSIONS

In this paper we derived a dispersion relation for a plate in supersonic flow with the boundary layer over the plate taken into account. For "moderate" wave lengths an analytical solution $\omega(k)$ is given and influence of the boundary layer on the plate stability is analysed. For short waves, when the Rayleigh equation cannot be simplified and solved analytically, numerical study is conducted. It is shown that the term of the order of k^2 in the Rayleigh equation yields additional destabilisation for a certain segment of the boundary layer thicknesses. For δ lying outside this segment, this term stabilizes the short waves.

REFERENCES

[1] Kramer, M. O. 1960 Boundary layer stabilization by distributed damping. *J. Amer. Soc. Nav. Engrs.* **72**, 25.

[2] Carpenter, P. W., Garrad, A. D. 1985 The hydrodynamic stability of flow over Kramer-type compliant surfaces: Part 1. Tollmien-Schlichting instabilities. *J. Fluid Mech.* **155**, 465–510.

[3] Carpenter, P. W., Garrad, A. D. 1986 The hydrodynamic stability of flow over Kramer-type compliant surfaces: Part 2. Flow-induced surface instabilities. *J. Fluid Mech.* **170**, 199–232.

[4] Lingwood, R. J., Peake, N. 1999 On the causal behaviour of flow over an elastic wall. *J. Fluid Mech.* **396**, 319–344.

[5] Miles, J. 2001 Stability of inviscid shear flow over a flexible boundary. *J. Fluid Mech.* **434**, 371–378.

[6] Reutov, V. P., Rybushkina, G. V. 1998 Hydroelastic instability threshold in a turbulent boundary layer over a compliant coating. *Phys. Fluids* **10(2)**, 417–425.

[7] Savenkov, I. V. 1995 The suppression of the growth of nonlinear wave packets by the elasticity of the surface around which flow occurs. *Comput. Math. Math. Phys.* **35(1)**, 73–79.

[8] Wiplier, O., Ehrenstein, U. 2001 On the absolute instability in a boundary-layer flow with compliant coatings. *European J. Mech. – B/Fluids* **20(1)**, 127–144.

[9] Boiko, A. V., Kulik, V. M. 2012 Stability of flat plate boundary layer over monolithic viscoelastic coatings. *Doklady Physics*, **57(7)**, 285–287.

[10] Bolotin, V. V. 1963 *Nonconservative problems of the theory of elastic stability*. Pergamon Press.

[11] Dowell, E. H. 1974 *Aeroelasticity of plates and shells*. Noordhoff International Publishing.

[12] Vedeneev, V. V. 2005 Flutter of a wide strip plate in a supersonic gas flow. *Fluid Dynamics* **5**, 805–817.

[13] Vedeneev, V. V., Guvernyuk, S. V., Zubkov, A. F., Kolotnikov, M. E. 2010 Experimental observation of single mode panel flutter in supersonic gas flow. *J. Fluids Struct.* **26**, 764–779.

[14] Vedeneev, V. V. 2012 Panel flutter at low supersonic speeds. *J. Fluids Struct.* **29**, 79–96.

[15] Vedeneev, V. V. 2013 Effect of damping on flutter of simply supported and clamped panels at low supersonic speeds. *J. Fluids Struct.* **40**, 366–372.

[16] Dowell, E. H. 1971 Generalized aerodynamic forces on a flexible plate undergoing transient motion in a shear flow with an application to panel flutter. *AIAA J.* **9(5)**, 834–841.

[17] Dowell, E. H. 1973 Aerodynamic boundary layer effect on flutter and damping of plates. *J. Aircraft* **10(12)**, 734–738.

[18] Hashimoto, A., Aoyama, T., Nakamura, Y. 2009 Effect of turbulent boundary layer on panel flutter. *AIAA J.* **47(12)**, 2785–2791.

[19] Gaspers, P. A., Jr., Muhlstein, L., Jr., Petroff, D. N. 1970 Further results on the influence of the turbulent boundary layer on panel flutter. NASA TN D-5798.

[20] Muhlstein, L., Jr., Gaspers, P. A., Jr., Riddle, D. W. 1968 An experimental study of the influence of the turbulent boundary layer on panel flutter. NASA TN D-4486.

[21] Miles, J. W. 1959 On panel flutter in the presence of a boundary layer. *J. Aero/Space Sciences* **26(2)**, 81–93, 107.

[22] Lees, L., Lin, C. C. 1946 Investigation of the stability of the laminar boundary layer in a compressible fluid. NACA TN 1115.

[23] Drazin, P. G., Reid, W. H. 2004 *Hydrodynamic Stability*. Cambridge University Press.

[24] Kornecki, A. 1979 Aeroelastic and hydroelastic instabilities of infinitely long plates. II. *Solid Mech. Archives* **4(4)**, 241–346.

[25] Vedeneev, V. V. 2013 Interaction of panel flutter with inviscid boundary layer instability in supersonic flow. *J. Fluid Mech.* **736**, 216–249.

[26] Schlichting, H. 1960 *Boundary Layer Theory*. McGraw-Hill.